Game Theory

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Roadmap: Game Theory – Static Games

A game is a multi-person decision problem. Strategic thinking in such environments requires players to guess the behavior of others to decide on the best course of action.

The analysis of static considers two different classes of games:

- Complete Information Static Games
- Incomplete Information Static Games (Not discussed!)

Introduces fundamental notions of strategy and best response.

Different Solution Concepts are presented:

- Dominant Strategy Equilibrium
- Iterative Elimination of Dominated Strategies
- Nash Equilibrium
- Bayes Nash Equilibrium (Not discussed!).

Section 1

Strategic Form Games:
Static Complete Information Games

Summary

Games of Complete Information:

- Definitions:
  - Game: Players, Actions, Payoffs
  - Strategy: Pure and Mixed
  - Best Response
- Solution Concept:
  - Dominant Strategy Equilibrium
  - Nash Equilibrium
- Properties of Nash Equilibria:
  - Non-Existence and Multiplicity
  - Inefficiency
- Examples

Introduction to Games

Any environment in which the choices of an individual affect the well being of others can be modeled as a game.

What pins down a specific game:

- Who participates in a game [Players]
- The choices that participants have [Choices]
- The well being of individuals [Payoffs]
- The information that individuals have [Rules of the Game]
- The timing of events and decisions [Rules of the Game]

Complete Information (Strategic Form) Game

A complete information game G consists of:

- A set of players:
  - N of size n
- An action set for each player in the game:
  - A_i for player i's
  - An action profile $\mathbf{a} = (a_1, a_2, ..., a_n)$ picks an action for each player
- A utility map for each player mapping action profiles to payoffs:
  - $u_i(\mathbf{a})$ denotes player i's payoff of action profile $\mathbf{a}$

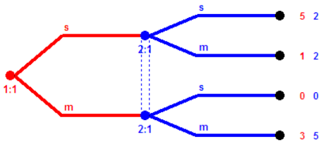
B/G	s	m
s	5,2	1,2
m	0,0	3,5

Representing Simultaneous Move Complete Info Games

- Strategic Form

1/2	s	m
s	5,2	1,2
m	0,0	3,5

- Extensive Form



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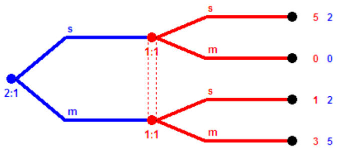

Feasible Payoffs and Efficiency

- Strategic Form

1/2	s	m
s	5,2	1,2
m	0,0	3,5

- Feasible Payoffs

Efficient payoffs are on the north-east boundary.

Information and Pure Strategies

A strategy in a game:

- is a map from information into actions
- it defines a plan of action for a player

In a complete information strategic form game:

- players have no private information
- players act simultaneously

In this context a strategy is any element of the set of actions

For instance a (pure) strategy for player i is simply $a_i \in A_i$

Notation

Define a profile of actions chosen by all players other than i by $\mathbf{a}_{-i}$:

$$\mathbf{a}_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$$

Define the set of possible action profiles for all players other that i as

$$A_{-i} = \times_{j \in N \setminus i} A_j$$

Define the set of possible action profiles for all players as

$$A = \times_{j \in N} A_j$$

Best Responses

The best response correspondence of player i is defined by:

$$b_i(\mathbf{a}_{-i}) = \arg\max_{a_i \in A_i} u_i(a_i, \mathbf{a}_{-i}) \text{ for any } \mathbf{a}_{-i} \in A_{-i}$$

Thus a_i is a best response to $\mathbf{a}_{-i}$ – i.e. $a_i \in b_i(\mathbf{a}_{-i})$ – if and only if:

$$u_i(a_i, \mathbf{a}_{-i}) \geq u_i(a_i', \mathbf{a}_{-i}) \text{ for any } a_i' \in A_i$$

BR identifies the optimal action for a player given choices made by others

For instance:

B/G	s	m
s	5,2	1,2
m	0,0	3,5

Section 2

Strategic Form Games: Dominance

Strict Dominance

- Strategy a_i **strictly dominates** a_i' if:

$$u_i(a_i, \mathbf{a}_{-i}) > u_i(a_i', \mathbf{a}_{-i}) \text{ for any } \mathbf{a}_{-i} \in A_{-i}$$

- a_i is **strictly dominant** if it strictly dominates any other a_i'
- a_i is **strictly undominated** if no strategy strictly dominates a_i
- a_i is **strictly dominated** (SDS) if a strategy strictly dominates a_i

In the following example s is strictly dominant for B:

B/G	s	m
s	5,-	2,-
m	0,-	1,-

Weak Dominance

- Strategy a_i **weakly dominates** a_i' if:

$$u_i(a_i, \mathbf{a}_{-i}) \geq u_i(a_i', \mathbf{a}_{-i}) \text{ for any } \mathbf{a}_{-i}$$

$$u_i(a_i, \mathbf{a}_{-i}) > u_i(a_i', \mathbf{a}_{-i}) \text{ for some } \mathbf{a}_{-i}$$

- a_i is **weakly dominant** if it weakly dominates any other a_i'
- a_i is **weakly undominated** if no strategy weakly dominates a_i
- a_i is **weakly dominated** (WDS) if a strategy weakly dominates a_i

In the following example s is weakly dominant for B:

B/G	s	m
s	5,-	2,-
m	0,-	2,-

Dominant Strategy Equilibrium

Definitions (Dominant Strategy Equilibrium DSE)

A strict Dominant Strategy equilibrium of a game G consists of a strategy profile $\mathbf{a}$ such that for any $\mathbf{a}'_{-i} \in A_{-i}$ and $i \in N$:

$$u_i(a_i, \mathbf{a}'_{-i}) > u_i(a'_i, \mathbf{a}'_{-i}) \text{ for any } a'_i \in A_i$$

- For weak DSE change $>$ with $\geq \dots$
- a profile $\mathbf{a}$ is a DSE **iff** $a_i \in b_i(\mathbf{a}'_{-i})$ for any $\mathbf{a}'_{-i}$ and $i \in N$
- Example (Prisoner's Dilemma):

A/B	N	C
N	5,5	0,6
C	6,0	1,1

Example: The Hotelling Game

There are two parties are in an election – "Left" and "Right".

Political view points are represented by a number between -1 and 1.

Voter views are uniformly distributed on [-1, 1].

The Hotelling Game:

- Each party chooses a policy position in the interval [-1, 1]:
  - The Right party chooses in [0, 1];
  - The Left party chooses in [-1, 0].
- Voters cast ballots in favor of the party closest to their ideal point.
- The party that gets a majority of votes wins.
- Parties care only about winning.

Where will each party position itself? Is there any dominant strategy?

Iterative Elimination of Dominated Strategies

If there is no DSE, we can still eliminate strictly dominated strategies, ... and by repeating the process we may rule out more strategies.

Consider the following example:

1\2	L	C	R
T	1,0	2,1	3,0
M	2,3	3,2	2,1
D	0,2	1,2	2,5

$\Rightarrow$

1\2	L	C	R
T	1,0	2,1	3,0
M	2,3	3,2	2,1
D	0,2	1,2	2,5

At the first instance only D is dominated for player 1.

No strategy is dominated a priori for player 2.

Strategies in green in the table are SDS and thus eliminated.

Iterative Elimination of Dominated Strategies

Once D has been eliminated from the game:

Strategy R is dominated for player 2.

No strategy is dominated for player 1.

1\2	L	C	R
T	1,0	2,1	3,0
M	2,3	3,2	2,1
D	0,2	1,2	2,5

$\Rightarrow$

1\2	L	C	R
T	1,0	2,1	3,0
M	2,3	3,2	2,1
D	0,2	1,2	2,5

Once R has been eliminated from the game:

Strategy T is dominated for player 1.

A final iteration yields (M, L) as the only surviving strategies.

Common Knowledge of Rationality

Definition

The fact F is a common knowledge if

- Every player knows F;
- Every player knows that every other player knows F;
- Every player knows that every player knows that every player knows F and so on.

Theorem

If, among players, there is a common knowledge of the game and of the fact that all players are rational, then the outcome of the game must be among those that survive iterative elimination of dominated strategies.

Strategies that survive iterative elimination of SDS are said to be **rationalizable**.

Dominance: Final Considerations I

Dominance and rationalizability are benchmarks of rationality.

The benefits of strict iterative elimination are that:

- the order of elimination is irrelevant;
- there is no need to know the other player's action;
- it all comes from rationality.

The limitations of strict iterative elimination are that:

- won't always reach solution;
- we must assume common knowledge of rationality and of the game;
- it often leads to inefficient outcomes.

Dominance: Final Considerations II

This is problematic in two ways:

- rationality must hold with probability 1;
- there must be unlimited "depth" of rationality.

Problem I: Rationality with Probability 1

Consider the following game for k very large:

1/2	L	R
U	2,5	3,4
M	0,-k	2,k

Iterative elimination leads for (U,L).

But is it likely player 2 will never play R for any value of k?

Dominance: Final Considerations III

Problem II: Unlimited Depth of Rationality

Consider a game in which:

- 10 players compete;
- each player writes a number between 1 & 1000;
- denote by $X = 1/2$ of the average of the players numbers;
- the player whose number is closest to X wins.

After iterative elimination of SDS, everyone quotes number 1.

But this does not match up with the empirical evidence.

A weaker notion of equilibrium may bypass some of these limitations.

Section 3

Strategic Form Games: Nash Equilibrium

Nash Equilibrium: Introduction

Dominance was the appropriate solution concept if players had no information or beliefs about choices made by others

The weaker notion of equilibrium that will be introduced presumes that:

- players have correct beliefs about choices made by others
- players choices are optimal given such beliefs
- the environment is common knowledge among players

Such model allows for tighter predictions when dominance has no bite

Nash Equilibrium

Definition (Nash Equilibrium NE)

A (pure strategy) Nash equilibrium of a game G consists of a strategy profile $\mathbf{a} = (a_i, \mathbf{a}_{-i})$ such that for any $i \in N$:

$$u_i(\mathbf{a}) \geq u_i(a_i', \mathbf{a}_{-i}) \text{ for any } a_i' \in A_i$$

- a profile $\mathbf{a}$ is a NE **iff** $a_i \in b_i(\mathbf{a}_{-i})$ for any $i \in N$

Properties:

- Strategy profiles are independent
- Strategy profiles common knowledge
- Strategies maximize utility given beliefs

Examples, Properties and Limitations

Games may have more than one NEs (Battle of the Sexes):

B\G	s	m
s	5,2	1,2
m	0,0	3,5

Nash equilibria may not be efficient (Prisoner's Dilemma):

A\B	N	C
N	5,5	0,6
C	6,0	1,1

Pure strategy Nash equilibria may not exist (Matching Pennies):

B\G	H	T
H	0,2	2,0
T	2,0	0,2

Games may have multiple NE (Battle of the Sexes):

B\G	s	m
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T	2,0	0,2

Example: Coordination Games

Consider a game played by two hunters, for $k \in [1, 8]$:

B\G	Stag	Hare
Stag	9,9	0,k
Hare	k,0	k,k

There are two NE:

- in one the hunters coordinate on hunting the stag;
- in the other they split and each hunts a hare on his own.

When compared to the hare-NE, the stag-NE entails:

- greater gains from coordination;
- greater risks of miscoordination.

If players communicate, they are likely to opt for Pareto dominating NE.

Nash equilibria can be viewed as **focal points**.

Example: Three Players

- A game with more than 2 players:

3	L		R	
1\2	A	B	A	B
T	1,1,0	0,0,0	0,1,1	0,2,1
D	0,1,1	1,2,0	1,0,0	2,1,1

- To find all NE check best reply maps:

3	L		R	
1\2	A	B	A	B
T	1,1,0	0,0,0	0,1,1	0,2,1
D	0,1,1	1,2,0	1,0,0	2,1,1

War of Attrition Example

Consider a game with two competitors involved in a fight:

- The set of players is $N = \{1, 2\}$.
- Competitors choose how much effort to put in a fight $A_i = [0, \infty)$.
- The value of winning the fight for competitor $i \in N$ is v_i.
- The highest effort wins the fight and ties are broken at random.
- For each competitor the cost of fighting is simply $\min\{a_i, a_j\}$.
- The payoff of competitor i given their effort levels thus satisfy:

$$u_i(a_i, a_j) = \begin{cases} v_i - a_j & \text{if } \ a_i > a_j \\ v_i/2 - a_j & \text{if } \ a_i = a_j \\ -a_i & \text{if } \ a_i < a_j \end{cases}$$

War of Attrition Example

As payoffs amount to:

$$u_i(a_i, a_j) = \begin{cases} v_i - a_j & \text{if } a_i > a_j \\ v_i/2 - a_j & \text{if } a_i = a_j \\ -a_i & \text{if } a_i < a_j \end{cases}$$

Best response functions satisfy:

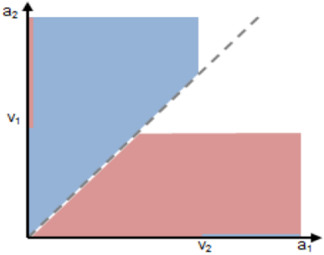
$$b_i(a_j) = \begin{cases} a_i > a_j & \text{if } a_j < v_i \\ a_i = 0 \text{ or } a_i > a_j & \text{if } a_j = v_i \\ a_i = 0 & \text{if } a_j > v_i \end{cases}$$

All Nash Equilibria of the game satisfy one of the following:

- $a_1 = 0$ and $a_2 \geq v_1$
- $a_2 = 0$ and $a_1 \geq v_2$

War of Attrition Example

An easy way to find the NE in such games is plotting BRs:



All Nash Equilibria of the game satisfy one of the following:

- $a_1 = 0$ and $a_2 \geq v_1$
- $a_2 = 0$ and $a_1 \geq v_2$

Example: Lobbying

Consider a game with two lobbyists petitioning over two versions of a bill:

- The value of having bill $i \in \{1, 2\}$ approved is v_i for i and zero for j.
- Lobbyists choose how many resources to invest $a_i \in [0, \infty)$.
- The probability that the policy preferred by lobbyist i is approved is

$$p_i = \frac{a_i}{a_i + a_j}$$

- The payoff of lobbyist i thus amounts to

$$u_i = \frac{a_i}{a_i + a_j} v_i - a_i$$

- The payoff is concave and single peaked for any value a_j.

Example: Lobbying

Taking first order conditions yields

$$\frac{a_j}{(a_i + a_j)^2} v_i = 1 \iff a_i + a_j = (v_i a_j)^{1/2}$$

Thus, best response function necessarily satisfy

$$b_i(a_j) = \begin{cases} (v_i^{1/2} - a_j^{1/2}) a_j^{1/2} & \text{if } a_j < v_i \\ 0 & \text{if } a_j \geq v_i \end{cases}$$

Solving the two shows that the unique NE necessarily satisfies

$$a_i = v_j \left[\frac{v_i}{v_i + v_j} \right]^2$$

DSE implies NE

Fact

Any dominant strategy equilibrium is a Nash equilibrium

Proof.

If $\mathbf{a}$ is a DSE then $a_i \in b_i(\mathbf{a}'_{-i})$ for any $\mathbf{a}'_{-i}$ and $i \in N$.

Which implies $\mathbf{a}$ is NE since $a_i \in b_i(\mathbf{a}_{-i})$ for any $i \in N$. ■

Nash Equilibrium: Final Considerations I

The most notable limitations of the NE solution concept are that:

- players have **correct beliefs** about other players' strategies;
- players are **rational** and choose the optimal action given these beliefs;
- NE are only immune to deviations by individual players, not groups.

The first requirement is very problematic, especially in games with more than one equilibrium.

Nash Equilibrium: Final Considerations II

NE is a good solution concept for:

- norms – everybody knows what side to drive on the road;
- preliminary talks – reaching an agreement that is self-enforcing;
- stable solutions played over time – the game is played many times; players choose the action the best based on past experience (disregarding any strategic consideration); NE is a necessary condition for stability (but not sufficient and not necessarily easy converge to).

Section 4

Strategic Form Games: Mixed Strategies

Introduction to Mixed Strategies

A problematic aspect of the solution concepts discussed in the first two lectures was that equilibria did not always exist.

Intuitively, existence was not guaranteed because players had no device to conceal their behavior from others.

Formally, the reasons for the lack of existence were:

- Non-convexities in the choice sets
- Discontinuities of the best response correspondences

Next we introduce mixed strategies which solve both problems and guarantee existence of at least a Nash equilibrium.

Mixed Strategy Definition

Consider complete information static game $\{N, \{A_i, u_i\}_{i \in N}\}$

A mixed strategy for $i \in N$ is a probability distribution over actions in A_i

Thus σ_i is a mixed strategy if:

- $\sigma_i(a_i) \geq 0$ for any $a_i \in A_i$
- $\sum_{a_i \in A_i} \sigma_i(a_i) = 1$

Intuitively $\sigma_i(a_i)$ is the probability that player i chooses to play a_i

E.G. $\sigma_1(B) = 0.3$ and $\sigma_1(C) = 0.7$ is a mixed strategy for 1 in:

1\2	B	C
B	2,0	0,2
C	0,1	1,0

Simplex

The set of possible probability distributions on a finite set B is:

- called **simplex**
- and denoted by $\Delta(B)$

It is straightforward to show that the simplex:

- Closed
- Bounded
- Convex

Notation

Define a profile of strategies for all players other than i by:

$$\boldsymbol{\sigma}_{-\boldsymbol{i}} = (\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_N)$$

Define a profile of strategies for all players by:

$$\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_N)$$

Payoffs from Mixed Strategies

Mixed strategy are chosen **independently** by players. Thus,

$$Pr(\mathbf{a}|\boldsymbol{\sigma}) = \prod_{j \in N} Pr(a_j|\boldsymbol{\sigma}) = \prod_{j \in N} \sigma_j(a_j)$$

The expected utility of player i from a mixed strategy profile σ is

$$u_i(\boldsymbol{\sigma}) = \sum_{\mathbf{a} \in \mathbf{A}} Pr(\mathbf{a}|\boldsymbol{\sigma}) u_i(\mathbf{a}) = \\ = \sum_{\mathbf{a} \in \mathbf{A}} \prod_{j \in N} \sigma_j(a_j) u_i(\mathbf{a})$$

Example: If players follow $\sigma_1(B) = \sigma_2(B) = 0.3$ in the game:

Payoffs:

1\2	B	C
B	2,0	0,2
C	0,1	1,0

Probabilities:

1\2	B	C
B	9%	21%
C	21%	49%

The payoff to player 1 is: $u_1(\sigma_1, \sigma_2) = (.09)2 + (.49)1 + (.42)0 = 0.67$

Best Reply Correspondences

Denote the best reply correspondence of i by $b_i(\boldsymbol{\sigma}_{-i})$

The map is defined by:

$$b_i(\boldsymbol{\sigma}_{-i}) = \arg\max_{\sigma_i \in \Delta(A_i)} u_i(\sigma_i, \boldsymbol{\sigma}_{-i})$$

For instance consider the game:

1\2	s	m
s	5,2	1,2
m	0,0	3,5

If $\sigma_1(s) = 1$ then any $\sigma_2(s) \in [0,1]$ satisfies $\sigma_2 \in b_2(\sigma_1)$

If $\sigma_1(s) < 1$ then only $\sigma_2(s) = 0$ satisfies $\sigma_2 \in b_2(\sigma_1)$

Dominated Strategies

- Strategy σ_i **weakly dominates** a_i if:

$$u_i(\sigma_i, \mathbf{a}_{-i}) \geq u_i(a_i, \mathbf{a}_{-i}) \text{ for any } \mathbf{a}_{-i}$$

$$u_i(\sigma_i, \mathbf{a}_{-i}) > u_i(a_i, \mathbf{a}_{-i}) \text{ for some } \mathbf{a}_{-i}$$

- a_i is **weakly undominated** if no strategy weakly dominates it
- This allows us to rule out more strategies than before, eg:

1\2	L	C	R
T	6,6	0,2	0,0
B	0,0	0,2	6,6

- $\sigma_2(L) = \sigma_2(R) = 0.5$ strictly dominates C since:

$$u_2(\sigma_2, a_1) = 3 > u_2(C, a_1) = 2$$

Nash Equilibrium

Definition (Nash Equilibrium NE)

A Nash equilibrium of a game consists of a strategy profile $\boldsymbol{\sigma} = (\sigma_i, \boldsymbol{\sigma}_{-i})$ such that for any $i \in N$:

$$u_i(\boldsymbol{\sigma}) \geq u_i(a_i, \boldsymbol{\sigma}_{-i}) \text{ for any } a_i \in A_i$$

Implicit to the definition of NE are the following assumptions:

- Each agent chooses his mixed strategy **independently** of others
- Each agent **knows** exactly **the strategies** the others adopt
- Each agent chooses so to **maximize expected utility** given his beliefs

Nash Equilibrium: Computation Help

- A strategy profile σ is a Nash Equilibrium if and only if:

$$u_i(\boldsymbol{\sigma}) = u_i(a_i, \boldsymbol{\sigma}_{-i}) \text{ for any } a_i \text{ such that } \sigma_i(a_i) > 0$$

$$u_i(\boldsymbol{\sigma}) \geq u_i(a_i, \boldsymbol{\sigma}_{-i}) \text{ for any } a_i \text{ such that } \sigma_i(a_i) = 0$$

- These conditions are evocative of complementary slackness as:

$$[u_i(\boldsymbol{\sigma}) - u_i(a_i, \boldsymbol{\sigma}_{-i})]\sigma_i(a_i) = 0,$$

$$u_i(\boldsymbol{\sigma}) - u_i(a_i, \boldsymbol{\sigma}_{-i}) \geq 0, \quad \sigma_i(a_i) \geq 0$$

- If a_i is strictly dominated, then $\sigma_i(a_i) = 0$ in any Nash equilibrium
- If a_i is weakly dominated, then $\sigma_i(a_i) > 0$ in a Nash equilibrium only if any profile of actions $\mathbf{a}_{-i}$ for which a_i is strictly worse occurs with zero probability

Games may have more than one NEs (Battle of the Sexes):

1\2	s	m
s	5,2	1,1
m	0,0	2,5

There are 2 PNE & a mixed NE in which $\sigma_1(s) = 5/6$ & $\sigma_2(s) = 1/6$:

$$u_1(s, \sigma_2) = 5\sigma_2(s) + (1 - \sigma_2(s)) = 2(1 - \sigma_2(s)) = u_1(m, \sigma_2)$$

$$u_2(m, \sigma_1) = \sigma_1(s) + 5(1 - \sigma_1(s)) = 2\sigma_1(s) = u_2(s, \sigma_1)$$

Games may have only mixed NE (Matching Pennies):

ST\GK	L	R
L	-1,+1	+1,-1
R	+1,-1	-1,+1

There is a unique NE in which $\sigma_1(L) = \sigma_2(L) = 1/2$:

$$-\sigma_i(L) + (1 - \sigma_i(L)) = \sigma_i(L) - (1 - \sigma_i(L))$$

Example: A Continuum of NE

Games may have a continuum of NE:

1\2	L	C	R
T	4,1	0,2	3,2
D	0,2	2,0	1,0

The game has 1 PNE & a continuum of NEs, namely:

$$\sigma_1(T) = 2/3 \;\&\; \sigma_2(C) = 1 - \sigma_2(L) - \sigma_2(R)$$

$$\sigma_2(L) = 1/3 - (2/3)\sigma_2(R) \;\text{ for }\; \sigma_2(R) \in [0, 1/2]$$

These are all NE's since:

$$u_2(L, \sigma_1) = \sigma_1(T) + 2(1 - \sigma_1(T)) = 2\sigma_1(T) = u_2(C, \sigma_1) = u_2(R, \sigma_1)$$

$$u_1(T, \sigma_2) = 4\sigma_2(L) + 3\sigma_2(R) = 2\sigma_2(C) + \sigma_2(R) = u_1(D, \sigma_2)$$