Uncertainty

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Roadmap: Choice under Uncertainty

We introduce the classical model of decision making under uncertainty.

This model is an essential benchmark to analyze behavior in numerous markets (finance, insurance, betting) and in any type of game.

The analysis proceeds as follows:

- Lotteries & Expected Utility.
- Axioms on Preferences.
- Representation Theorem.
- Applications to Insurance.
- Attitudes towards Risk.
- Experimental Evidence.
- New Theories.

Section 1

Decision Theory: Uncertainty

Uncertainty

There are N possible **outcomes** (or states).

Denote the **set of possible** outcomes by

$$X = (x_1, ..., x_N).$$

A **simple lottery** L is a probability distribution $(p_1, ..., p_N)$ over outcomes in which:

- p_i denotes the probability of any outcome $i \in \{1, ..., N\}$.

A **compound lottery** $(\alpha^1 L^1, ..., \alpha^k L^k)$ is a probability distribution over simple lotteries in which:

- L^j denotes a simple lottery for any $j \in \{1, ..., k\}$;
- α^j denotes the probability of lottery L^j.

From Compound to Simple Lotteries

Consider any compound lottery $(\alpha^1 L^1, ..., \alpha^k L^k)$.

Let p_i^j is the probability of outcome i in lottery L^j.

If so, there exists a corresponding **reduced simple lottery** $(p_1, ..., p_N)$ st:

$$p_i = \sum_{j=1}^{k} \alpha^j p_i^j, \quad \text{for any } i = 1, ..., N.$$

Example: Consider the following scenario:

- $X = (\$1, \$2)$,
- $L_1 = (1/2, 1/2)$,
- $L_2 = (1, 0)$.

For a compound lottery $(0.5L_1, 0.5L_2)$,

The corresponding reduced lottery is $(0.75, 0.25)$.

Axioms: Completeness, Transitivity, Continuity

Assume that decision makers regard any compound lottery as equivalent to its reduced simple lottery.

The preferences of the decision maker are defined over the set of simple lotteries $\mathcal{L}$, and denoted by $\succeq$.

Preferences satisfy **continuity** if for any $L, L', L'' \in \mathcal{L}$, the following two sets are closed:

$$\{\alpha \in [0,1] : (\alpha L', (1-\alpha)L'') \succeq L\} \text{ and } \{\alpha \in [0,1] : L \succeq (\alpha L', (1-\alpha)L'')\}.$$

The following two axioms on preferences are invoked throughout:
[A1] $\succeq$ is complete and transitive.
[A2] $\succeq$ satisfies continuity.

A1 and A2 imply that preferences can be represented by a utility function.

Axioms: Independence

A third and final axiom known as the **independence axiom** is needed to derive a classical and fundamental representation theorem:
[A3] $\succeq$ satisfies independence.

Preferences satisfy **independence** if for any $L, L', L'' \in \mathcal{L}$ and $\alpha \in [0,1]$:

$$L \succeq L' \text{ if and only if } (\alpha L, (1-\alpha)L'') \succeq (\alpha L', (1-\alpha)L'').$$

Example: $L = 100\$$, $L' = 10\$$, $L'' =$ Trip to Paris.

A3 implies that $L \succeq L'$ if and only if

$$(\alpha L, (1-\alpha)[\text{Trip}]) \succeq (\alpha L', (1-\alpha)[\text{Trip}]).$$

The utility of from the trip cancels out. A decision maker should thus keep preferring more money to less regardless of whether she wins the trip.

Representation Theorem

Theorem (Representation Theorem)

If A1-A3 hold, there exists a vector of utilities $(u_1, ..., u_N)$ such that, for any two lotteries $L = (p_1, ..., p_N)$ and $L' = (p'_1, ..., p'_N)$,

$$L \succeq L' \text{ if and only if } \sum_{i=1}^{N} p_i u_i \geq \sum_{i=1}^{N} p'_i u_i.$$

Furthermore, any other vector of utilities $(\bar{u}_1, ..., \bar{u}_N)$ represents the same preferences if and only if for some constants $b > 0$ and a:

$$\bar{u}_i = a + bu_i.$$

The key features of the representation imply that:

- preferences are linear in probabilities;
- the utility of a lottery is the **expected utility** of the outcomes;
- only affine transformations of utility represent the same preferences.

Intuition: Independence Axiom

Consider the following scenario:

- $X = (x_1, x_2, x_3)$,
- $L = (1, 0, 0)$,
- $L' = (0, 1, 0)$,
- $L'' = (0, 0, 1)$.

Note that the representation theorem implies

$$(\alpha L, (1-\alpha)L'') \succeq (\alpha L', (1-\alpha)L'') \textbf{ iff } \alpha u_1 + (1-\alpha)u_3 \geq \alpha u_2 + (1-\alpha)u_3.$$

This in turn immediately implies the independence axiom, as

$$\alpha u_1 + (1-\alpha)u_3 \geq \alpha u_2 + (1-\alpha)u_3 \textbf{ iff } u_1 \geq u_2 \textbf{ iff } L \succeq L'.$$

Section 2

Risk Attitudes

Attitudes towards Risk

Now assume that outcomes in X are quantifiable so that $X = \mathbb{R}_+$.
This is a good assumption for money, consumption, wealth...

A lottery now consists of a cumulative distribution

$$F : \mathbb{R}_+ \to [0, 1],$$

where $F(x)$ is the probability that the outcome is smaller or equal to x.
The preferences have an expected utility representation

$$U(F) = \int_X u(x) dF(x),$$

where $u(x)$ is the utility of x with certainty.
If the density $f(.)$ is well defined, then

$$U(F) = \int_X u(x) f(x) dx.$$

Assume that $u(.)$ is strictly **increasing and continuous**.

Attitudes towards Risk

A decision maker is **risk averse** if and only if, for any lottery F,

$$\int_X u(x)dF(x) \leq u\left(\int_X xdF(x)\right).$$

A decision maker is **risk neutral** if and only if, for any lottery F,

$$\int_X u(x)dF(x) = u\left(\int_X xdF(x)\right).$$

A decision maker is **risk loving** if and only if, for any lottery F ,

$$\int_X u(x)dF(x) \geq u\left(\int_X xdF(x)\right).$$

Fact

A decision maker is risk averse if and only if $u(.)$ is concave.

A decision maker is risk loving if and only if $u(.)$ is convex.

Attitudes towards Risk

A decision maker is **strictly risk averse** if and only if, for any lottery F that does not assign probability equal to one to the outcome $\int_X x dF(x)$,

$$\int_X u(x) dF(x) < u\left(\int_X x dF(x)\right).$$

A decision maker is **strictly risk loving** if and only if, for any lottery F that does not assign probability equal to one to the outcome $\int_X x dF(x)$,

$$\int_X u(x) dF(x) > u\left(\int_X x dF(x)\right).$$

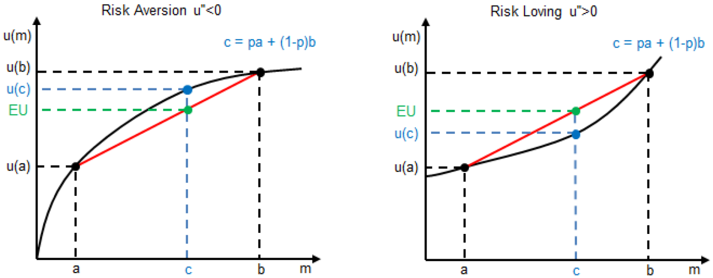
Examples:

- strictly risk averse, $u(x) = \sqrt{x}$;
- risk neutral, $u(x) = x$;
- strictly risk loving, $u(x) = x^2$.

Attitudes towards Risk

Consider a lottery $X = \{a, b\}$ and $L = \{p, 1 - p\}$.

If so, graphically observe that:



Example: Insurance

Consider the following scenario:

- M denotes the initial wealth;
- L denotes a loss that may be incurred;
- p denotes the probability of the loss;
- S denotes the amount of coverage;
- r denotes unit price of coverage (unit premium).

Two outcomes are possible: $\begin{cases} M - rS & \text{with probability} 1 - p \\ M - L + S - rS & \text{with probability} p \end{cases}$.

A risk-averse consumer chooses $S \leq L$ to maximize expected utility,

$$(1-p)u(M - rS) + pu(M - L + (1-r)S).$$

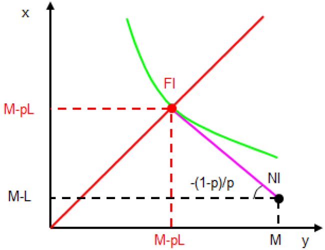
The FOC immediately yields

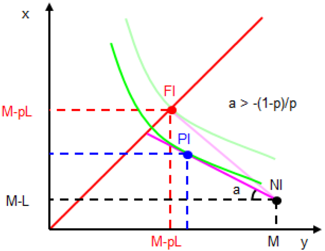
$$r(1-p)u'(M - rS) = (1-r)pu'(M - L + (1-r)S).$$

Exercise: Show that, if $r = p$, a risk averse consumer chooses $S = L$.

Example: Insurance

Demand for insurance can be derived exactly as demand for other goods:





The consumer partially insures whenever the price exceeds the risk $r > p$.

Measures of Risk Aversion

Intuitively, the more concave is a utility function the more risk averse is the decision maker.

Since $u(.)$ and $bu(.)$, $b > 0$, represent the same preferences, $u''(.)$ is not a satisfactory measure of risk aversion.

Define the **coefficient of absolute risk aversion** as:

$$r_A(x) = -\frac{u''(x)}{u'(x)}.$$

Example: Consider preferences $u(x) = -e^{-ax}$.

If so, $r_A(x) = -\frac{-a^2 e^{-ax}}{ae^{-ax}} = a$

Concave Transformations I

Consider two decision makers, one with utility $u(.)$, one with utility $v(.)$.

Both $u(.)$ and $v(.)$ are strictly increasing.

Therefore there exists $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = \phi(v(x)) \text{ for any } x \in \mathbb{R}_+.$$

It follows that

$$u'(x) = \phi'(v(x))v'(x) \text{ and}$$
$$u''(x) = \phi''(v(x))(v'(x))^2 + \phi'(v(x))v''(x).$$

By eliminating $\phi'(v(x))$ from the two expressions obtain

$$u''(x) = \phi''(v(x))(v'(x))^2 + \frac{u'(x)}{v'(x)}v''(x).$$

Concave Transformations II

The previous expression is equivalent to

$$\phi''(v(x)) = \frac{u''(x)v'(x) - v''(x)u'(x)}{(v'(x))^3}.$$

Hence, $\phi(.)$ is concave if and only if

$$-\frac{v''(x)}{v'(x)} \leq -\frac{u''(x)}{u'(x)}$$

Thus, $u(.)$ is a **concave transformation** of $v(.)$ if and only if $u(.)$ has a larger coefficient of absolute risk aversion than $v(.)$.

Absolute risk aversion is a **partial order** (not all preferences can be compared) over preferences that compares ranks preferences based on attitudes towards risk.

Relative Risk Aversion

Absolute risk aversion measures risk attitudes towards absolute changes in wealth at the current wealth level.

A different measure risk aversion can capture risk attitudes towards percentage changes in wealth at the current wealth level.

Define the **coefficient of relative risk aversion** as:

$$r_A(x) = -\frac{xu''(x)}{u'(x)}.$$

Relative risk aversion is a **partial order** (not all preferences can be compared) over preferences that compares ranks preferences based on risk attitudes for percentage changes in wealth.

Aside: Decreasing absolute risk aversion has been used as a justification for income inequality. . .

Section 3

Paradoxes and Advances: Choice under Uncertainty

Allais Paradox I

Consider outcome set $X = \{5 \text{ millions}, 1 \text{ millions}, 0 \text{ millions }\}$.

First consider the following two lotteries:

Outcome:	5 M$	1 M$	0 M$
Lottery A:	0%	100%	0%
Lottery B:	10%	89%	1%

Evidence suggests individuals choose A over B.

Such behavior requires

$$u(1) > 0.1u(5) + 0.89u(1) + 0.01u(0),$$

$$\Rightarrow 0.11u(1) > 0.1u(5) + 0.01u(0).$$

Allais Paradox II

Next consider the following two lotteries:

Outcome:	5 M$	1 M$	0 M$
Lottery C:	0%	11%	89%
Lottery D:	10%	0%	90%

Evidence suggests individuals choose D over C.

Such behavior requires

$$0.11u(1) + 0.89u(0) < 0.1u(5) + 0.9u(0),$$

$$\Rightarrow 0.11u(1) < 0.1u(5) + 0.01u(0).$$

Preferences violate the independence axiom (not linear in probabilities)!

In experiments individuals will correct this mistake over time.

Possible explanation comes from regret theory (choices are made to avoid disappointment from an outcome that did not materialize).

Subjective Probabilities

We have assumed that a decision maker ranks explicit and objective probability distributions over outcomes.

However, probabilities may be implicit and subjective.

For example, the decision maker may be asked to choose between:
(a) 100£ if Arsenal wins the Champions League;
(b) 100£ if Barcelona wins the Champions League.

Subjective probabilities may be revealed by the choices of decision makers.

If (a) is preferred to (b), we may conclude that the decision maker thinks that the probability of Arsenal winning the title is higher than the probability of Barcelona winning the title.

Ellsberg Paradox & Ambiguity Aversion

But choices may not consistent with "probabilistic" assessments.

Consider an urn containing 100 balls.

Some balls are red, while the rest are blue.

The decision maker is **not told** how many are red and how many are blue.

Extract a ball from this urn and consider the bets:
(a) £ 100 if the ball is red;
(b) £ 100 if the ball is blue;
(c) £ 99 with probability 50% - coin toss.

Many individuals would choose (c).

However, either $\Pr(\textit{red}) \geq 1/2$ or $\Pr(\textit{blue}) \geq 1/2$, regardless of whether probabilities are objective or subjective.

Maxmin Utility (Gilboa & Schmeidler)

Let S denote the set of possible states.

A **bet** is a function $x : S \longrightarrow \mathbb{R}_+$.

In particular, $x(s)$ is the monetary outcome in state s.

The set of "possible" probability distributions over the set of states is C.

Under reasonable axioms, preferences over bets x can always be represented by a utility function

$$\min_{F \in C} \int_S u(x(s)) dF(s).$$

This approach can explain Ellsberg's Paradox. When individuals are pessimistic about their beliefs, they may prefer an objective lottery.

Limits to Risk-Aversion

Risk-aversion is a standard economic paradigm, but it imposes to much discipline when comparing **small** risks to **large** ones.

Rabin (2000) shows that:

- if an EU maximizer rejects a 50-50 gamble to win 11 or lose 10 **regardless of his wealth**, w,
- he rejects a 50-50 gamble to win y or lose 100, for any value of y!!

To show this observe that concavity implies:

$$u(w) - u(w-10) < +10u'(w-10)$$

$$u(w) - u(w+11) < -11u'(w+11)$$

Since he turns down the gamble, we also know:

$$u(w) > 0.5u(w-10) + 0.5u(w+11)$$

Combining all three inequalities yields:

$$u'(w+11) - u'(w-10) < -\frac{1}{11}u'(w-10)$$

Limits to Risk-Aversion

So gaining 21 causes the marginal utility of money to fall by 9%. Iterating this logic shows that small stakes compound quickly and cause marginal utility to plummet on large stakes.

According to EU, there is no amount we could offer him to accept the chance of losing 100.

EU was not designed to explain small-stakes gambles and requires individuals to be locally risk neutral.

Evidence suggests that the vast majority of individuals would reject the small bet, Barberis, Huang and Thaler (2006) and Arrow (1971).

A solution involves kinking the utility function around a reference point.

Prospect Theory (Kahneman & Tversky)

Subjective probabilities are **biased**:

- small probabilities are overestimated,
- large probabilities are underestimated.

The utility of an outcome may depend on a **reference point** r.

Let $(x_1, ..., x_N)$ denote a set of monetary outcomes.

Consider a lottery over outcomes $(p_1, ..., p_N)$.

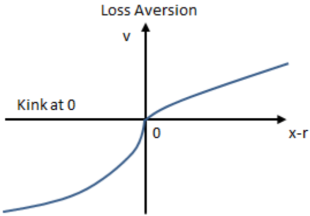
Let $d_i(p_1, ..., p_N)$ denote the distorted probability of outcome x_i.

If so, under plausible axioms preferences can be represented as,

$$\sum_{i=1}^{N} d_i(p_1, ..., p_N)u(x_i|r).$$

Loss Aversion

If r is wealth and $u(x_i|r) = v(x_i - r)$, utility is defined on gains & losses.



Loss Aversion: Example

Suppose an agent has a piecewise-linear value function:

$$v(x) = \begin{cases} x & \text{if} \quad x \geq 0 \\ 2x & \text{if} \quad x < 0 \end{cases}$$

Let the reference point be her initial wealth $r = w$.

Does she accept a 50/50 gamble to win 11 or lose 10?

- If she turns it down, she gets:

$$v(w - r) = v(0) = 0$$

- If she accepts, she gets:

$$\frac{v(w + 11 - r) + v(w - 10 - r)}{2} = \frac{v(11) + v(-10)}{2} = -4.5$$

- So she turns it down

Loss Aversion: Example

What is the smallest value y for which she accepts a 50/50 gamble to win y or lose 100?

- If she turns it down, she gets:

$$v(w-r) = v(0) = 0$$

- If she accepts, she gets:

$$\frac{v(w+y-r)+v(w-100-r)}{2} = \frac{v(y)+v(-100)}{2} = \frac{y}{2} - 100$$

- So she will accept if $y \geq 200$.

Loss aversion also provides an explanation for the endowment effect.

Gambles on Gains and Losses

Most people are:

- risk-averse when thinking only of gains but
- risk-loving when thinking of losses.

Let (y, p) be a gamble that gives:

- y with probability p and
- 0 with probability $(1 - p)$.

Kahneman & Tversky 1979 shows that:

(4000,0.8) 20%	(3000,1) 80 %	(-4000,0.8) 92%	(-3000,1) 8%
(3000,0.9) 86%	(6000,0.45) 14%	(-3000,0.9) 8%	(-6000,0.45) 92%

Rejects equal proportions choosing the risky option.

Diminishing Sensitivity

The second finding of Kahneman & Tversky 1979 is *diminishing sensitivity*.

Marginal sensitivity to changes from the reference point are smaller the farther the change is from the reference point.

KT argue this is also an extension of how we perceive things. For example, an object 101 feet away is indistinguishable from an object 100 feet away, but an object 2 feet away is easily perceived as very different from an object 1 foot away. Likewise, the relative impact of receiving (resp. losing) 10 instead of 0 is larger than the impact of receiving (resp. losing) 1,010 instead of 1,000.

This is not just risk-aversion: people will be risk-averse in the domain of gains but risk-seeking in the domain of losses.

Properties of Value Functions

Value Function is assumed to be:

1. $v(x)$ is continuous for any x and twice differentiable for $x \neq 0$.
2. $v(x)$ is strictly increasing, and $v(0) = 0$.
3. $v(y) + v(-y) < v(x) + v(-x)$ for $y > x > 0$.
4. $v''(x) \leq 0$ for x > 0, and $v''(x) \geq 0$ for $x < 0$.
5. $v'_-(0)/v'_+(0) > 1$.

These theories are convenient analytically and valuable guides to behavior. But occasionally fail to provide a unified approach to capture all biases displayed by human behavior.