General Equilibrium

Sepehr Ekbatani
(TeIAS)

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Roadmap: General Equilibrium & Markets

The aim is to introduce a general model of behavior of consumers and firms that captures the consequences of spillovers across markets.

Since markets are interdependent, extending the theory of partial equilibrium to general equilibrium is a natural step.

The analysis proceeds as follows:

- Exchange Economies.
- Definition of a Walrasian Equilibrium.
- Properties of Walrasian Equilibria.
- Efficiency & Pareto Efficiency.
- Welfare Theorems.
- General Equilibrium with Production.

Section 1

Exchange Economies

Exchange Economies

First we abstract from production and study how goods are traded by consumers endowed of a fixed amounts.

Consider an **exchange economy**:

- with n commodities and k consumers;
- in which each consumer j is described by:
  - a utility function $U^j(x^j) = U^j\left(x_1^j, ..., x_n^j\right)$;
  - an initial endowment $\omega^j = \left(\omega_1^j, ..., \omega_n^j\right)$.

A **consumption bundle** for consumer j is $x^j = \left(x_1^j, ..., x_n^j\right)$.

An **allocation** $x = (x^1, ..., x^k)$ assigns a bundle to each consumer.
An allocation x is **feasible** if $\sum_{j=1}^k x^j \leq \sum_{j=1}^k \omega^j$.
A feasible allocation obtains by redistributing or destroying endowments.

Walrasian Equilibrium

Definition

A **Walrasian equilibrium** $(\bar{x}, \bar{p})$ consists of an allocation $\bar{x} = (\bar{x}^1, ..., \bar{x}^k)$ and of a price vector $\bar{p} = (\bar{p}_1, ..., \bar{p}_n)$ such that;

1. $\bar{x}$ is feasible;
2. $\bar{x}^j$ maximizes $U^j(x^j)$ subject to $\bar{p}x^j = \bar{p}\omega^j$ for all $j = 1, ..., k$.

Walrasian equilibria are also referred to as competitive or general equilibria.

A competitive equilibrium consists of an allocation and of a price vector st:

- consumers choices are optimal when taking as given such prices;
- the implied demands are feasible (i.e. all markets clear).

Introductory Example: Market Clearing

Consider a simple economy with:

- $k = 2$ and $n = 2$;
- $\omega^1 = (3, 1)$ and $U^1(x^1) = \min\{x_1^1, x_2^1\}$;
- $\omega^2 = (1, 3)$ and $U^2(x^2) = x_1^2 x_2^2$.

Solving the consumers' problems we obtain that

$$x_1^1(p, p\omega^1) = \frac{3p_1 + p_2}{p_1 + p_2} \qquad x_2^1(p, p\omega^1) = \frac{3p_1 + p_2}{p_1 + p_2}$$
$$x_1^2(p, p\omega^2) = \frac{p_1 + 3p_2}{2p_1} \qquad x_2^2(p, p\omega^2) = \frac{p_1 + 3p_2}{2p_2}$$

Equilibrium conditions require the two markets to clear:

$$x_1^1 + x_1^2 = \frac{3p_1 + p_2}{p_1 + p_2} + \frac{p_1 + 3p_2}{2p_1} = 4;$$
$$x_2^1 + x_2^2 = \frac{3p_1 + p_2}{p_1 + p_2} + \frac{p_1 + 3p_2}{2p_2} = 4.$$

Introductory Example: Equilibrium

From the equilibrium condition for good 1 obtain

$$\frac{(p_1 + 3p_2)(p_2 - p_1)}{2p_1(p_1 + p_2)} = 0 \quad \Rightarrow \quad p_1 = p_2.$$

Substituting in the condition for the market for good 2

$$2 + 2 = 4.$$

Key observations:

- Equilibrium in the market for good 1 implies
  Equilibrium in the market for good 2.
- The equilibrium price vector is not unique.
  Only the price ratio is pinned down, $\bar{p}_1/\bar{p}_2 = 1$.
- The equilibrium allocation satisfies:
  $x_1^1 = 2, x_2^1 = 2, x_1^2 = 2$, and $x_2^2 = 2$.

Introductory Example: Edgeworth Box

A useful exercise is to plot the example in an Edgeworth Box.

Proceed by:

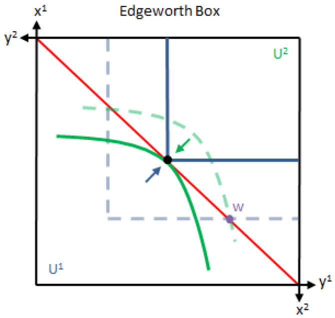
- plotting the Edgeworth box and the feasible allocations;
- finding the endowment, and drawing budget constraints;
- plotting the equilibrium itself;
- showing that $p_1/p_2 > 1$ cannot be an equilibrium, since consumer 1 demand of good 2 is more than 2 wants to give.

Show more generally that an equilibrium is a price ratio for which consumers's demands are optimal and markets clear.

At an interior solution consumers equalize MRS. But boundary solutions, multiplicity of equilibria and non-existence are other possible phenomena.

Introductory Example: Edgeworth Box

The plot of Edgeworth Box in this example thus amounts to:



The solution is interior and equalizes the MRS of the two consumers.

Normalizing Prices

Recall that:

- $x_i^j(p, p\omega^j)$ denotes j's demand for good i;
- $x^j(p, p\omega^j)$ denotes vector of j's demands;
- $x_i^j(p, p\omega^j)$ is homogeneous of degree 0 in p.

Fact

Because demands are homogeneous of degree 0 if $\bar{p}$ is an equilibrium price vector, $t\bar{p}, t > 0$, is also an equilibrium price vector.

Hence, we can normalize one price.

Without loss, set $\bar{p}_i = 1$ for one arbitrary good i.

Call good i the **numeraire good**.

Excess Demand & Walras Law

Define the vector of j's **excess demands** as

$$z^j(p, p\omega^j) = x^j(p, p\omega^j) - \omega^j.$$

Define the vector of **aggregate excess demands** as

$$z(p) = \sum_{j=1}^{k} z^j(p, p\omega^j).$$

Define the **aggregate excess demand for good** i as

$$z_i(p) = \sum_{j=1}^{k} \left(x_i^j(p, p\omega^j) - \omega_i^j \right).$$

Lemma (Walras Law)

If LNS holds, the value of excess demand is equal to zero, $p_i z_i(p) = 0$.

Walras Law: Proof

Proof: As LNS (Local Non Satiation) holds,

$$pz^j(p, p\omega^j) = p\left(x^j(p, p\omega^j) - \omega^j\right) = 0, \text{ for any } j = 1, ..., k.$$

Summing over all players then implies that

$$pz(p) = p\sum_{j=1}^{k} z^j(p, p\omega^j) = \sum_{j=1}^{k} pz^j(p, p\omega^j) = 0.$$

The latter can be restated in terms of goods as follows

$$pz(p) = \sum_{i=1}^{n} p_i \sum_{j=1}^{k} \left(x_i^j(p, p\omega^j) - \omega_i^j\right) = \sum_{i=1}^{n} p_i z_i(p) = 0.$$

Observe that $p_i z_i(p) \leq 0$ for all i since:

- prices are non-negative, $p_i \geq 0$;
- excess demands are non-positive by feasibility $z_i(p) \leq 0$.

But the latter than implies that $p_i z_i(p) = 0$ for all i!

Some Remarks: Assumptions

Competitive equilibria are reduced descriptions of an economy in which institutions of exchange remain unmodelled (a market can be conceived as an auctioneer, as a central market place, or as a market maker).

We do not explain how consumers trade. Agents act as price-taking utility-maximizing machines, markets are conceived as frictionless institutions in which exchange takes place at once (**invisible hand**).

Some strong assumption were invoked to derive our results.
These required markets to be frictionless, and required consumers:

- to be able to buy and sell any amount of goods at current prices;
- to be aware of the current prices;
- to take prices as given;
- to face the same prices;
- to maximize utility.

Some Remarks: Limitations & Evidence

Limitations arise because the model cannot capture environments in which:

- goods trade at different prices at different locations;
- consumers ignore, but dynamically forecast of future prices;
- consumers cannot buy or sell quantities without affecting your prices.

State contingent commodities can bypass some of these limitations.

Evidence: In controlled experiments, subjects were induced to have different preferences, and put into an exchange economy, buying and selling commodities. A middle man who ignored the aggregate demand and supply, was chosen to clear markets buying from sellers, while selling to buyers. This operation was repeated numerous times to allow learning about the environment. Subjects learnt quickly the prices at which markets would clear, and these prices were close to equilibrium prices. The evidence supported the Walrasian Equilibrium as an appropriate solution concept for large centralized markets (Plott 1986).

Section 2

Efficiency and Welfare

Pareto Efficiency

A common justification for markets is that markets attain efficiency.

The next definition is crucial for the welfare properties of equilibria.

Definition

An allocation $\bar{x} = (\bar{x}^1, ..., \bar{x}^k)$ is **Pareto efficient** if and only if:

1. it is feasible;
2. there is no other feasible allocation $\hat{x} = (\hat{x}^1, ..., \hat{x}^k)$ such that:
   (a) $U^j(\hat{x}^j) \geq U^j(\bar{x}^j)$ for any $j \in \{1, ..., k\}$;
   (b) $U^j(\hat{x}^j) > U^j(\bar{x}^j)$ for some $j \in \{1, ..., k\}$.

Comments:

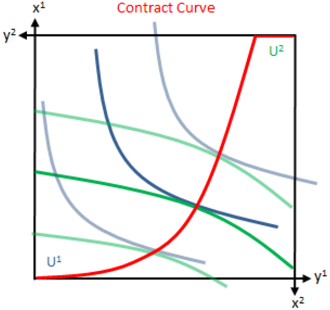
- there are no prices involved in the definition of Pareto efficiency;
- in an EB: find the inefficient, efficient, boundary efficient allocations.

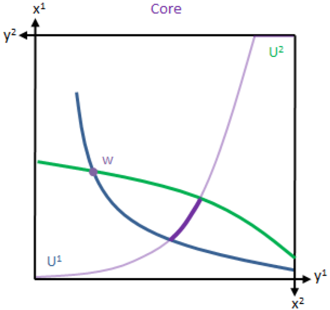
The Pareto Problem & The Core

A Pareto efficient (PE) allocation $(\bar{x}^1, \bar{x}^2)$ maximizes $U^1(x^1)$ subject to:

$$U^2(x^2) = U^2(\bar{x}^2) \quad \text{and} \quad x^1 + x^2 = \omega^1 + \omega^2.$$

The **contract curve** identifies all PE allocations, while the **core** identifies those PE allocations in which players are better off than at the endowment.





Optimality in the Pareto Problem

If $(\bar{x}^1, \bar{x}^2)$ is interior, FOC for the Pareto problem imply

$$MRS^1 = MRS^2 \quad \& \quad \bar{x}^1 + \bar{x}^2 = \omega^1 + \omega^2.$$

Example: Consider and economy with:

- $k = 2$ and $n = 2$;
- $\omega^1 = (3, 1)$ and $U^1(x^1) = x_1^1 x_2^1$;
- $\omega^2 = (1, 3)$ and $U^2(x^2) = x_1^2 x_2^2$.

FOC for the Pareto problem require:

$$x_2^1/x_1^1 = x_2^2/x_1^2, \; x_1^1 + x_1^2 = 4, \text{ and } x_2^1 + x_2^2 = 4.$$

It follows immediately that

$$x_2^1/x_1^1 = (4 - x_2^1)/(4 - x_1^1) \Rightarrow x_2^1 = x_1^1.$$

The contract curve coincides with the 45^o line.

Example: Pareto Problem

Consider and economy with:

- $k = 2$ and $n = 2$;
- $\omega^1 = (0, 0)$ and $U^1(x^1) = x_1^1 x_2^1$;
- $\omega^2 = (4, 2)$ and $U^2(x^2) = x_1^2 + x_2^2$.

The interior KKT FOC for the Pareto problem require:

$$x_2^1/x_1^1 = 1,\ x_1^1 + x_1^2 = 4, \quad \text{and} \quad x_2^1 + x_2^2 = 2.$$

Such conditions cannot be satisfied when $U^1(x^1) > 4$.

Looking at corners in this scenario, we find that $x_2^1 = 2$.

Thus, the contract curve can be expressed as

$$x_2^1 = \min\{x_1^1, 2\}.$$

The First Welfare Theorem

Theorem (The First Welfare Theorem)

If LNS holds, any competitive equilibrium allocation is Pareto efficient.

Proof: Let $(\bar{x}, \bar{p})$ be a competitive equilibrium.

By definition of competitive equilibrium, $\bar{x}$ is feasible.

Suppose however that $\bar{x}$ is not Pareto efficient.

If so, there exists another feasible allocation $\hat{x}$ such that

(a) $U^j(\hat{x}^j) \geq U^j(\bar{x}^j)$ for any $j \in \{1, ..., k\}$;

(b) $U^j(\hat{x}^j) > U^j(\bar{x}^j)$ for some $j \in \{1, ..., k\}$.

The First Welfare Theorem

Proof Continued: Recall that $\hat{x}$ satisfies

(a) $U^j(\hat{x}^j) \geq U^j(\bar{x}^j)$ for any $j \in \{1, ..., k\}$;

(b) $U^j(\hat{x}^j) > U^j(\bar{x}^j)$ for some $j \in \{1, ..., k\}$.

First, observe that (a) implies that $\bar{p}\hat{x}^j \geq \bar{p}\omega^j$ for any j.

If $\bar{p}\hat{x}^j < \bar{p}\omega^j$ for some j, by LNS there exists $\tilde{x}^j$ such that

$$\bar{p}\tilde{x}^j < \bar{p}\omega^j \text{ and } U^j(\tilde{x}^j) > U^j(\hat{x}^j) \geq U^j(\bar{x}^j).$$

But this is impossible as $\bar{x}^j$ maximizes utility.

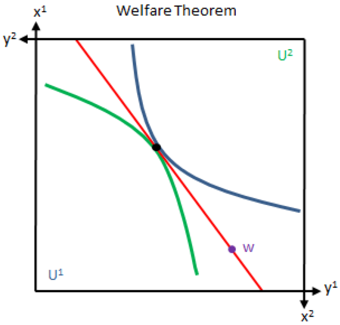
Next observe that (b) implies that $\bar{p}\hat{x}^j > \bar{p}\omega^j$ for some j.
This follows, as $\bar{x}^j$ maximizes $U^j(x^j)$ subject to $\bar{p}x^j \leq \bar{p}\omega^j$.

Hence, $\sum_{j=1}^k \bar{p}\hat{x}^j > \sum_{j=1}^k \bar{p}\omega^j$, contradicting the feasibility of $\hat{x}$. QED

First Welfare Theorem: Plot

Key assumptions:

- Price taking behavior [agents take prices as a given]
- No externalities [utility only depends on consumption]
- Complete markets [any commodity can be traded]



Comments on the First Welfare Theorem

Interior competitive equilibria are always Pareto efficient as consumers set their MRS equal to the price ratio. Therefore, MRS coincide across players and efficiency obtains.

General Comments on Pareto Efficiency:

- Although Pareto Efficiency is a meaningful concept to evaluate welfare it completely abstracts from fairness. A dictator consuming all goods would yield Pareto efficiency. Other welfare criteria have been contemplated such as max-min utility, or utility aggregation.
- The Jungle Economics: instead of prices, it is possible to consider power relations. An equilibrium allocation is such that no one can be better off by taking something available to him. This model also has an equilibrium which is Pareto efficient.

Example Externalities

Consider an economy with:

- $k = 2$ and $n = 2$;
- $\omega^1 = (3, 1)$ and $U^1(x^1) = \min\{x_1^1, x_2^1\} - 2x_1^2$;
- $\omega^2 = (1, 3)$ and $U^2(x^2) = x_1^2 x_2^2 - 2x_2^1$.

The equilibrium allocation again satisfies:

- $x_1^1 = 2,\ x_2^1 = 2,\ x_1^2 = 2,\ x_2^2 = 2.$

Both consumers are worse off after trade.

The equilibrium is Pareto dominated.

The initial utilities are $U^1 = -1, U^2 = 1$.

Equilibrium utilities are $U^1 = -2, U^2 = 0$.

The Second Welfare Theorem (Easy)

Theorem (The Second Theorem of Welfare - Easy)

If $\bar{x}$ is Pareto optimal then:

there exists a price vector $\bar{p} \in \mathbb{R}^n_+$ such that $(\bar{p}, \bar{x})$ is a competitive equilibrium.

Proof: To prove this we need a version of the separating hyperplane theorem...

The Second Welfare Theorem (Medium)

Theorem (The Second Theorem of Welfare - Medium)

Consider a Pareto efficient allocation $\bar{x}$.

Suppose that a competitive equilibrium $(\hat{p}, \hat{x})$ exists when $\omega = \bar{x}$.

If so, $(\hat{p}, \bar{x})$ is a competitive equilibrium.

Proof: Observe that for any consumer $j = 1, ..., k$:

- since $\bar{x}^j$ is in the budget constraint of j, $U^j(\hat{x}^j) \geq U^j(\bar{x}^j)$;
- since $\bar{x}$ is Pareto efficient, $U^j(\hat{x}^j) = U^j(\bar{x}^j)$.

Hence, $\bar{x}^j$ is optimal given $\hat{p}$.

The Second Welfare Theorem (Hard)

Theorem (The Second Theorem of Welfare - Hard)

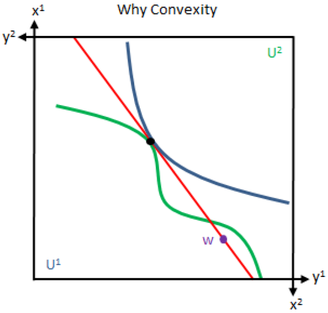
Consider a Pareto efficient allocation $\bar{x}$ and suppose that:

1. The allocation $\bar{x}$ is interior.
2. The preferences of every consumer are
   - convex,
   - continuous,
   - strictly monotonic.

Then, $\bar{x}$ is a competitive equilibrium allocation when $\omega = \bar{x}$.

Convexity

The importance of convex preferences is detailed by the plot below:



Consumer 2 does not maximize utility at ω.

If preferences are not convex, existence cannot be guaranteed.

Comments on the Second Welfare Theorem

The second welfare states that any efficient allocation can be decentralized into a competitive equilibrium if transfers are feasible.

To decentralize an efficient allocation planner would need:

- information about preferences and endowments;
- power to enact transfers;
- to make sure that all prices are known.

This exercise will fail however if there are: externalities, market power, public goods, or incomplete information.

Existence of a competitive equilibrium is proven via fixed point theorems, if preferences are continuous, convex and locally non-satiated.

GE is a closed interrelated system, as opposed to partial equilibrium.
It is suitable to address problems which relate to the whole economy.
Its beauty lies in the ambitious results obtained with few free parameters.

Section 3

Production Economies

General Equilibrium with Production

Consider an economy with:

- n commodities;
- k consumers;
- m firms.

Let $f^i : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ denote the production function of firm i.

Let each firm maximizes profits and denote the profit functions by

$$\pi^i(p) = \max_{x_i} pf^i(x^i) - px^i.$$

Let α^j_i be consumer j's share of the profits of firm i.
Consumer j's budget constraint becomes

$$px^j \leq p\omega^j + \sum_{i=1}^{m} \alpha^j_i \pi_i(p).$$

Walrasian Equilibrium with Production

Definition

A **Walrasian equilibrium** $(\bar{x}, \tilde{x}, \bar{p})$ consists of an allocation $(\bar{x}, \tilde{x}) \in \mathbb{R}_+^{(k+m)n}$ and of a price vector $\bar{p} \in \mathbb{R}_+^n$ such that:

1. $\tilde{x}^i$ solves for all $i = 1, ..., m$

$$max_{x_i} pf^i(x^i) - px^i;$$

2. $\bar{x}^j$ solves for all $j = 1, ..., k$

$$max_{x^j} U^j(x^j) \text{ st } px^j \leq p\omega^j + \sum_{i=1}^{m} \alpha_i^j \pi_i(p);$$

3. $(\tilde{x}, \bar{x})$ is feasible:

$$\sum_{j=1}^{k} \bar{x}^j + \sum_{i=1}^{m} \tilde{x}^i \leq \sum_{j=1}^{k} \omega^j + \sum_{i=1}^{m} f^i(\tilde{x}^i).$$

All the main results derived for exchange economies extend to production economies with minor modifications.

Example Production: Walrasian Equilibrium

Consider the following economy:

1. Two firms, 1 and 2, produce goods y_1 and y_2 using the same input x. Production functions satisfy

$$y_1 = \sqrt{x_1} \text{ and } y_2 = \sqrt{x_2},$$

   where x_i is the amount of x used for producing y_i.

2. Two consumers, A and B, have utility functions

$$U^A = y_1^A y_2^A \text{ and } U^B = y_1^B y_2^B.$$

   Consumers derive no utility from input x.

3. The total endowment of y_1 and y_2 is zero.
   Consumer A owns firm 1 and 5 units of x.
   Consumer B owns firm 2 and 3 units of x.

Example Production: Walrasian Equilibrium

Let p_i denote the price of output y_i, let r denote the price of input x.

Normalize $p_2 = 1$.

Firm profit maximization FOC require that

$$\frac{p_1}{2\sqrt{x_1}} - r = 0 \text{ and } \frac{1}{2\sqrt{x_2}} - r = 0.$$

Such conditions pin down factor demands,

$$x_1^S = \left(\frac{p_1}{2r}\right)^2 \text{ and } x_2^S = \left(\frac{1}{2r}\right)^2,$$

output supplies,

$$y_1^s = \frac{p_1}{2r} \text{ and } y_2^s = \frac{1}{2r},$$

and consequently profit functions,

$$\pi_1 = \frac{(p_1)^2}{4r} \text{ and } \pi_2 = \frac{1}{4r}.$$

Example Production: Walrasian Equilibrium

The firms' problems imply that:

- the income of consumer A is $m^A = 5r + (p_1)^2/4r$;
- the income of consumer B is $m^B = 3r + 1/4r$.

From FOC of the consumers' problems find output demands. For output 1,

$$y_1^A = \frac{1}{2p_1}\left(\frac{(p_1)^2}{4r} + 5r\right) \text{ and } y_1^B = \frac{1}{2p_1}\left(\frac{1}{4r} + 3r\right).$$

Then by Walras law, the equilibrium market clearing conditions are

$$y_1 \text{ market :} \quad \frac{1}{2p_1}\left(\frac{1+(p_1)^2}{4r} + 8r\right) = \frac{p_1}{2r}$$

$$x \text{ market :} \quad \left(\frac{p_1}{2r}\right)^2 + \left(\frac{1}{2r}\right)^2 = 8$$

The competitive equilibrium satisfies

$$p_1 = 1,\ r = 1/4,\ y_1^A = y_2^A = 9/8, \text{ and } y_1^B = y_2^B = 7/8.$$

Pareto Efficiency with Production

The definition of Pareto efficient allocation only changes to the extent that feasibility does.

Definition

An allocation $(\bar{x}, \tilde{x}) \in \mathbb{R}_+^{(k+m)n}$ is **Pareto efficient** if and only if:

1. $(\bar{x}, \tilde{x})$ is feasible:

$$\sum_{j=1}^{k} \bar{x}^j + \sum_{i=1}^{m} \tilde{x}^i \leq \sum_{j=1}^{k} \omega^j + \sum_{i=1}^{m} f^i(\tilde{x}^i);$$

2. there is no other feasible allocation $(\hat{x}, \check{x})$ such that:
   (a) $U^j(\hat{x}^j) \geq U^j(\bar{x}^j)$ for any $j \in \{1, ..., k\}$;
   (b) $U^j(\hat{x}^j) > U^j(\bar{x}^j)$ for some $j \in \{1, ..., k\}$.

Example Production: Pareto Efficiency

Extend the above example to general production functions

$$y_1 = f_1(x_1) \text{ and } y_2 = f_2(x_2).$$

Suppose that total input endowment is $x_1 + x_2 = \gamma$.

By allocating the input x between the two production processes, obtain the **production possibility frontier**

$$y_2 = T(y_1).$$

Formally, if $f_1^{-1}(y_1)$ denotes the inverse of $f_1(x_1)$, we have that

$$T(y_1) = f_2(\gamma - f_1^{-1}(y_1)).$$

Example Production: Pareto Efficiency

Recall that the marginal rate of transformation, MRT, is

$$\frac{dT}{dy_1} = -\frac{\partial f_2/\partial x}{\partial f_1/\partial x}$$

Pareto efficient allocations can now be found by choosing $(y_1^A, y_2^A, y_1^B, y_2^B)$ so to maximize

$$U^A(y_1^A, y_2^A) \text{ subject to}$$

$$U^B(y_1^B, y_2^B) = \bar{u} \text{ and } y_2^A + y_2^B = T\left(y_1^A + y_1^B\right).$$

FOC immediately yields that

$$MRS^A = MRS^B = MRT.$$

Concluding Comments

GE is a parsimonious and flexible theory, which relies on price taking, individual maximization, and market clearing to derive implications about market outcomes.

It can accommodate time, location and state contingent pricing once commodity spaces are enlarged (an empirically relevant extension).

Conclusions on welfare further require the absence of externalities and private information, and the completeness of markets.