Production and Firms

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Classical Theory of Production

Our unit of analysis in terms of behavior is a **Firm**.

We won't discuss the ownership of a firm, or its management and organization. We only ask how to operate the firm to maximize profits.

The firm is treated as a black box with no owners, workers or managers. Just a transition box that transforms inputs into outputs.

The main ingredients of the classical theory of production are:

- **Technology Constraints**
  (Limits to production due to technological or legal constraints);
- **Profit Maximizing Behavior**
  (Firms maximize a particular objective function, profits);
- **Free Entry & Price Taking Behavior**.

This theory imposes little discipline on technology, but lots on payoffs.
Firms could maximize sales, survival, welfare of workers and/or managers.

Roadmap: Production

The aim of the section is to introduce a simple model that allows us to predict the behavior of firms in the market.

The analysis proceeds as follows:

- Basic definitions, vocabulary and technological constraints.
- Firms' optimization problem:
  - profit maximization,
  - cost minimization,
  - duality.
- Supply, profit function and cost function.
- The short run and the long run.
- Partial equilibrium and taxes.
- Monopoly Pricing
- Criticism and alternative theories.

Section 1

Technology

The Technology

Definition

A **production plan** $z = (z_1, ..., z_m)$ is a vector of net outputs.

For a given production plan z, we say that:

- z_i is an **input** if $z_i < 0$;
- z_i is an **output** if $z_i > 0$.

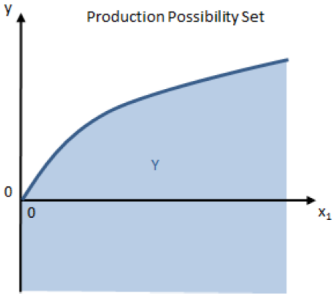
The **production possibility set**, Y, is the set of technologically feasible production plans.

Examples: Single Output

With only one output, we can write a production plan as $(y, -x)$ where:

- y denotes the amount of output and
- $x = (x_1, ...x_n)$ is a non-negative vector of n inputs.

For the case, of one input we have:



Properties of Y and Isoquants

We usually assume Y to be: non-empty, closed, and that can do nothing.

Although sunk costs or a positive level of inputs in place may change this.

Restrict the analysis to the case of **one** output.

The **Input Requirement Set** is defined as

$$V(y) = \{x \in \mathbb{R}^n \mid (y, -x) \in Y\}.$$

An **Isoquant** is defined as

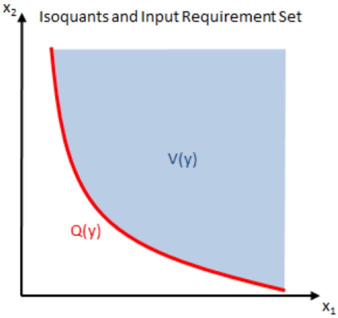
$$Q(y) = \{x \in \mathbb{R}^n \mid x \in V(y) \text{ and } x \notin V(y') \text{ if } y' > y\}.$$

Example: Isoquants

If $Y = \{(y, -x_1, -x_2) | y \leq \sqrt{x_1} + \sqrt{x_2}\}$, we have that:

$$V(\hat{y}) = \{(x_1, x_2) \mid \hat{y} \leq \sqrt{x_1} + \sqrt{x_2}\},$$

$$Q(\hat{y}) = \{(x_1, x_2) \mid \hat{y} = \sqrt{x_1} + \sqrt{x_2}\}.$$



The Production Function

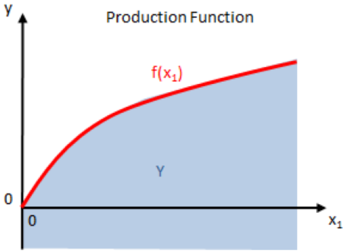
Definition

A **production function** is a map $y = f(x)$ where $f(x)$ is the maximum output obtainable from x in Y.

There are close connections between f and the properties of Y.

We can either make restrictions on Y or on f.

With 1 output and n inputs, Y is convex $\Rightarrow$ f is concave.



Example: Cobb-Douglas Production Function

Consider the following production function,

$$y = x_1^{\alpha} x_2^{\beta}.$$

The equation describing its isoquant for $y = 2$ is

$$2 = x_1^{\alpha} x_2^{\beta} \Longrightarrow x_2 = \left(\frac{2}{x_1^{\alpha}}\right)^{\frac{1}{\beta}}.$$

The slope of this isoquant is

$$\frac{dx_2}{dx_1} = -\frac{\alpha}{\beta}\left(\frac{2}{x_1^{\alpha+\beta}}\right)^{\frac{1}{\beta}}$$

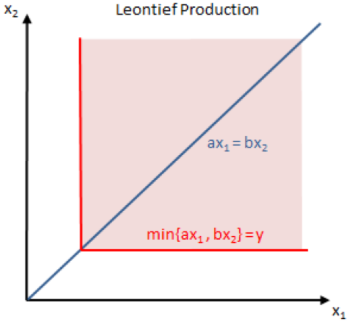
It expresses the trade-off between x_1 and x_2 that keeps output constant.

Example: Leontief Production Function

Consider the following production function,

$$y = min\{ax_1, bx_2\}.$$

Its isoquants are non-differentiable and plotted below



Technical Rate of Substitution

The slope of isoquants is found by totally differentiating the production function and evaluating such derivative at $dy = 0$.

Consider a differentiable production function $y = f(x_1, x_2)$.

Totally differentiating the production function we obtain

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2.$$

Setting $dy = 0$, we find the **Technical Rate of Substitution** (TRS),

$$\frac{dx_2}{dx_1} = -\frac{(\partial f / \partial x_1)}{(\partial f / \partial x_2)}$$

Technical Rate of Substitution

At a fixed output level y, the TRS expresses how much we need to change x_2 , if we had changed x_1, in order to keep output at y.

The TRS is the ratio of the marginal products,

$$MP_i = (\partial f / \partial x_i).$$

For the Cobb Douglas production function

$$TRS = -\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{\beta x_1^{\alpha} x_2^{\beta-1}} = -\frac{\alpha x_2}{\beta x_1}$$

Related: at fixed inputs x, the **Marginal Rate of Transformation** (MRT) expresses how much we need to change output y_1, if we had changed output y_2, in order to keep inputs at x.

Returns to Scale

A production function $f(x)$ can exhibit:

- Constant Returns to Scale (CRS), if

$$f(tx) = tf(x) \text{ for any } t > 0.$$

- Decreasing Returns to Scale (DRS), if

$$f(tx) < tf(x) \text{ for any } t > 1 \text{ and } x \text{ for which } f(x) > 0.$$

- Increasing Returns to Scale (IRS):

$$f(tx) > tf(x) \text{ for any } t > 1 \text{ and } x \text{ for which } f(x) > 0.$$

For instance, convexity of Y and $0 \in Y$ together imply non-increasing returns to scale.

Example: Linear Technology

Consider a linear production function,

$$y = ax_1 + bx_2.$$

The map displays CRS, since

$$f(tx) = atx_1 + btx_2 = t(ax_1 + bx_2) = tf(x).$$

Example: Cobb-Douglas Technology

For a Cobb Douglas production function $y = x_1^{\alpha} x_2^{\beta}$ we have,

$$f(tx) = (tx_1)^{\alpha}(tx_2)^{\beta} = t^{\alpha+\beta} x_1^{\alpha} x_2^{\beta}.$$

Therefore, the production function displays:

- CRS when $\alpha + \beta = 1$, since

$$f(tx) = t^{\alpha+\beta} x_1^{\alpha} x_2^{\beta} = t x_1^{\alpha} x_2^{\beta} = tf(x).$$

- DRS when $\alpha + \beta < 1$, since

$$f(tx) = t^{\alpha+\beta} x_1^{\alpha} x_2^{\beta} < t x_1^{\alpha} x_2^{\beta} = tf(x) \text{ for } t > 1 \ \& \ x_1^{\alpha} x_2^{\beta} \neq 0.$$

- IRS when $\alpha + \beta > 1$, since

$$f(tx) = t^{\alpha+\beta} x_1^{\alpha} x_2^{\beta} > t x_1^{\alpha} x_2^{\beta} = tf(x) \text{ for } t > 1 \ \& \ x_1^{\alpha} x_2^{\beta} \neq 0.$$

Homogeneity & Homoteticity

Definition

A function $f : \mathbb{R}^n_+ \to \mathbb{R}$ is **homogenous** of degree k if and only if

$$f(tx) = t^k f(x) \text{ for any } t > 0 \ \& \ x \in \mathbb{R}^n_+.$$

Example: $y = x_1^\alpha x_2^\beta$ is homogenous of degree $\alpha + \beta$.

Definition

A function $f : \mathbb{R}^n_+ \to \mathbb{R}$ is **homothetic** if a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ and a homogenous of degree 1 function $h : \mathbb{R}^n_+ \to \mathbb{R}$ exist such that

$$f(x) = g(h(x)) \text{ for any } x \in \mathbb{R}^n_+.$$

Exercise: If a function is homothetic, TRS at x equals TRS at tx.

Hint: If a map is homogeneous of degree k, its partial derivatives are?

Section 2

Profit Maximization

Intro to Profit Maximization

We have introduced assumptions about firms:

- Firms are profit maximizing, price taking entities.
- Firms have a technology defined by production possibilities set Y.
- We think of Y as non-empty, closed, and that can do nothing.
- The boundary of Y can be defined as the production function.
- The technology is what we make assumptions on.
- We have introduced concepts of returns to scale.

Now we turn to profit maximization:

- The highest profit line on Y will determine the solution.
- But a solution may not necessarily exist.
- For example if the production function exhibits IRS. GRAPH.
- For existence we need Y to be either strictly convex or bounded.

Profit Maximization

Firms are price-takers by **perfect competition**.

We employ the following conventions:

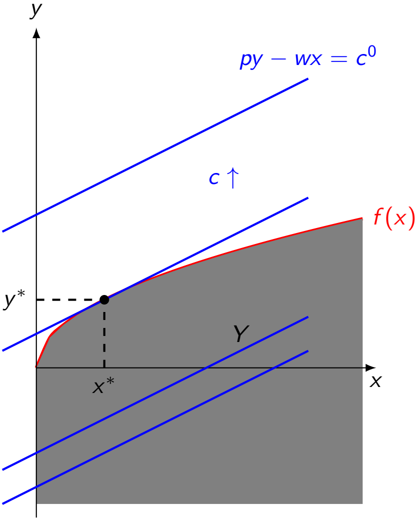
- y denotes output;
- p denotes the price of output;
- $x = (x_1, ..., x_n)$ denotes a vector of inputs;
- $w = (w_1, ..., w_n)$ denotes the vector of input prices;
- $wx = \sum_{i=1}^{n} w_i x_i$ denotes expenditure on inputs.

A firm chooses $(y, -x)$ to maximize

$$py - wx \quad \text{subject to} \quad (y, -x) \in Y.$$

The highest profit line intersecting Y identifies the solution if it exists.
But no solution may exist (e.g. if Y exhibits IRS).

Graphical Intuition



Profit Maximization

By profit maximization, a firm produces the maximal output given inputs.

Thus, the problem can be restated using the production function $f(x)$.

A firm chooses $(y, -x)$ to maximize

$$py - wx \quad \text{subject to} \quad y = f(x).$$

Hence, a firm chooses x to maximize

$$pf(x) - wx$$

So with single output the firm essentially chooses only inputs.

Input Demands & Output Supply

Solving the profit maximization problem, we obtain:

- the vector of (unconditional) **factor or input demands**

$$x(p, w) = (x_1(p, w), x_2(p, w), ..., x_n(p, w));$$

- the **supply function** for output

$$y(p, w) = f(x(p, w)).$$

Factor demands and the supply function are homogeneous of degree zero in (p, w),

$$x(p, w) = x(tp, tw) \text{ and } y(tp, tw) = y(p, w).$$

Do we always have a solution? No, for example CRS with some prices, or IRS. We need Y to be bounded from above and strictly convex.

First Order Necessary Conditions

Recall that $x(p, w)$ maximizes $pf(x) - wx$.

This is an constrained optimization problem with linear non-negativity constraints, and a possibly concave objective function.

If $x(p, w)$ is an interior solution, **First Order Conditions** require

$$p \left. \frac{\partial f(x)}{\partial x_i} \right|_{x=x(p,w)} - w_i = 0 \text{ for any } i = 1, 2, ..., n.$$

FOC for the profit maximization problem require:

- the marginal product of an input to equal its price normalized by the output price;
- the marginal rate of transformation between two inputs to equal their price ratio.

Second Order Sufficient Conditions

Define second order partial derivatives as follows,

$$f_{ij}(x) = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

Also define the Hessian matrix associated to the optimization problem as

$$H(x) = \begin{bmatrix} pf_{11}(x) & ... & pf_{1n}(x) \\ ... & ... & ... \\ pf_{n1}(x) & ... & pf_{nn}(x) \end{bmatrix}.$$

Necessary conditions also require $H(x)$ to be negative semi-definite at the solution $x = x(p, w)$.

If f is concave, or Y is strictly convex, FOC are sufficient for $x(p, w)$ to be a global maximum (not necessarily unique), as $H(x)$ is negative definite.

SOC only discipline f as the rest of the objective function is linear.

The Profit Function

Define the profit function $\pi(p, w)$ as

$$\pi(p, w) = pf(x(p, w)) - wx(p, w).$$

Recall that this is a maximum value function.

Example: Consider the production function $y = \sqrt{x_1} + \sqrt{x_2}$.

FOC require that:

$$\frac{1}{2}p(x_1)^{-\frac{1}{2}} - w_1 = 0 \Rightarrow x_1(p, w) = \left(\frac{p}{2w_1}\right)^2;$$

$$\frac{1}{2}p(x_2)^{-\frac{1}{2}} - w_2 = 0 \Rightarrow x_2(p, w) = \left(\frac{p}{2w_2}\right)^2.$$

Therefore, it follows that:

$$y(p, w) = \frac{p}{2w_1} + \frac{p}{2w_2} \;\;\&\;\; \pi(p, w) = \frac{p^2}{4}\left(\frac{1}{w_1} + \frac{1}{w_2}\right)$$

Profit Maximization Examples

Example: Consider the production function $y = \sqrt{min\{x_1, x_2\}}$.

First note that to maximize profits $x_1 = x_2$.

Hence, x_1 is chosen to maximize

$$p\sqrt{x_1} - (w_1 + w_2)x_1.$$

The unconditional factor demands are

$$x_1(p, w) = x_2(p, w) = \left(\frac{p}{2(w_1 + w_2)}\right)^2.$$

Therefore, it follows that:

$$y(p, w) = \frac{p}{2(w_1 + w_2)} \& \pi(p, w) = \frac{p^2}{4}\left(\frac{1}{w_1 + w_2}\right).$$

Profit Maximization Examples

Example: $y = min\{x_1, x_2\}$. If $p > w_1 + w_2$, profits are unbounded. The firm's supply is perfectly elastic at $p = w_1 + w_2$ (CRS).

Example: $y = x_1 + x_2$. If $p > min\{w_1, w_2\}$, profits are unbounded. The firm's supply is perfectly elastic at $p = min\{w_1, w_2\}$ (CRS).

Example: $y = (x_1 + x_2)^2$. The maximum profits are unbounded and the firm's supply is not well defined (IRS).

The previous examples establish that:

- we **don't always have a solution**;
- we **cannot always use first order conditions**;
- we sometimes have **multiple solutions**.

Properties of the Profit Function

Lemma

The profit function $\pi(p, w)$ satisfies the following properties

1. $\pi(p, w)$ is non-decreasing in p;
2. $\pi(p, w)$ is non-increasing in w_i , for any $i = 1, ..., n$;
3. $\pi(p, w)$ is homogeneous of degree 1 in (p, w);
4. $\pi(p, w)$ is convex in (p, w);
5. $\pi(p, w)$ is continuous in (p, w).

These are predictions that can be **tested**.

The intuition for the first two parts is obvious (plot to show).

Proving Properties of the Profit Function

1. Take $p' \geq p$. Then,

$$\pi(p, w) \leq p' y(p, w) - wx(p, w) \leq \pi(p', w).$$

2. The proof for w_i is analogous and left as an exercise.

3. By definition, for any $(y', -x') \in Y$ we have

$$py(p, w) - wx(p, w) \geq py' - wx'.$$

Hence, for $t > 0$ and for any $(y', -x') \in Y$,

$$tpy(p, w) - twx(p, w) \geq tpy' - twx'.$$

Thus, $(y(p, w), -x(p, w))$ is optimal at (tp, tw) and

$$\pi(tp, tw) = t\pi(p, w).$$

Proving Properties of the Profit Function

4. Consider two price vectors, (p, w) and (p', w').

For some $\lambda \in (0, 1)$ define:

$$(p'', w'') = \lambda(p, w) + (1 - \lambda)(p', w').$$

With these definitions, note that:

$$(1) \quad \pi(p, w) \geq py(p'', w'') - wx(p'', w'');$$

$$(2) \quad \pi(p', w') \geq p'y(p'', w'') - w'x(p'', w'').$$

Multiplying (1) by λ and (2) by $(1 - \lambda)$, and summing

$$\lambda\pi(p, w) + (1 - \lambda)\pi(p', w') \geq p''y(p'', w'') - w''x(p'', w''),$$

which implies: $\lambda\pi(p, w) + (1 - \lambda)\pi(p', w') \geq \pi(p'', w'')$.

5. Exercise.

Hotelling's Lemma

The profit function obtains by plugging $y(p, w)$ and $x(p, w)$ in profits,

$$\pi(p, w) = py(p, w) - wx(p, w).$$

Can we reverse the operation to recover $y(p, w)$ and $x(p, w)$?

The Envelope Theorem then implies the following result.

Lemma

Hotelling's Lemma states that

$$\frac{\partial \pi(p, w)}{\partial p} = y(p, w) \quad \& \quad \frac{\partial \pi(p, w)}{\partial w_i} = -x_i(p, w), \text{ for } i = 1, ..., n.$$

If Y is well behaved, price taking behavior and fixed prices imply a dual description of technology.

Such a description has a great virtue in applications, as we can compute supply directly from the envelope theorem.

Implications of Hotelling's Lemma

By Hotelling's Lemma and the convexity of the profit function we have

$$\frac{\partial y(p,w)}{\partial p} = \frac{\partial^2 \pi(p,w)}{\partial p^2} \geq 0$$

$$\frac{\partial x_i(p,w)}{\partial w_i} = -\frac{\partial^2 \pi(p,w)}{\partial w_i^2} \leq 0$$

This is a simple **Slutsky equation** with **testable predictions**.

Example: Consider the production function $y = \sqrt{x_1} + \sqrt{x_2}$.

Immediately obtain that

$$\pi(p,w) = \frac{p^2}{4}\left(\frac{1}{w_1} + \frac{1}{w_2}\right), \quad y(p,w) = \frac{\partial \pi(p,w)}{\partial p} = \frac{p}{2w_1} + \frac{p}{2w_2}, \ldots$$

Implications of Hotelling's Lemma

Thus, by convexity of the profit function it follows that:

- **own-price effects are positive for output**;
- **own-price effects are negative for inputs**.

As diagonal terms of a positive semi-definite matrix, are non-negative.
This is known as the **law of supplies**: quantities respond in the same direction as price changes. Generally it can be written as

$$(p - p')(y - y') \geq 0.$$

There are no wealth effects, just substitution effects since prices have no effect on constraints now.
Another testable implication is that

$$\frac{\partial x_j(p, w)}{\partial w_i} = \frac{\partial x_i(p, w)}{\partial w_j} = -\frac{\partial^2 \pi(p, w)}{\partial w_i \partial w_j}.$$

Cross-price effects are symmetric (not very intuitive, but testable).

Extra: Alternative Proof Strategy

We have shown by exploiting Hotelling that

$$\frac{\partial x_i(p,w)}{\partial w_i} < 0 \quad \& \quad \frac{\partial y(p,w)}{\partial p} > 0$$

To prove this directly, note that FOC require

$$p.Df(x(p,w)) = w.$$

By totally differentiating FOC wrt w_1, we obtain

$$p.D^2 f(x(p,w))\frac{\partial x(p,w)}{\partial w_1} = \begin{bmatrix} pf_{11} & \dots & pf_{1n} \\ & \vdots & \\ pf_{n1} & \dots & pf_{nn} \end{bmatrix} \begin{bmatrix} \partial x_1/\partial w_1 \\ \vdots \\ \partial x_n/\partial w_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Extra: Alternative Proof Strategy

$$\frac{\partial x_1}{\partial w_1} = \frac{\begin{vmatrix} 1 & pf_{12} & \dots & pf_{1n} \\ 0 & pf_{22} & \dots & pf_{2n} \\ & & \vdots & \\ 0 & pf_{n2} & \dots & pf_{nn} \end{vmatrix}}{|H|} = \frac{\begin{vmatrix} pf_{22} & \dots & pf_{2n} \\ & \vdots & \\ pf_{n2} & \dots & pf_{nn} \end{vmatrix}}{|H|}$$

Since the Hessian of f is negative semi-definite:

- the denominator is either zero or has the same sign as $(-1)^n$.
- the numerator is either zero or has the same sign as $(-1)^{n-1}$.

Hence, $\partial x_1/\partial w_1 \leq 0$, if H well defined so that $|H| \neq 0$.

Profit maximization $\Rightarrow$ input demands are non-increasing in own price.

We will derive this explicitly later. Remember that with consumers it depended on income and substitution effects.

Extra: Alternative Proof Strategy

To determine the sign of $\partial y/\partial p$, differentiate FOC wrt p

$$Df(x(p,w)) + p\frac{\partial x(p,w)}{\partial p}^T D^2 f(x(p,w)) = 0$$

$$\Rightarrow Df(x(p,w)) = -p\frac{\partial x(p,w)}{\partial p}^T D^2 f(x(p,w))$$

If so, with some manipulation, we find that

$$\frac{\partial y(p,w)}{\partial p} = \frac{\partial f(x(p,w))}{\partial p} = Df(x(p,w))\frac{\partial x(p,w)}{\partial p}$$

$$-p\frac{\partial x(p,w)}{\partial p}^T D^2 f(x(p,w))\frac{\partial x(p,w)}{\partial p} \geq 0$$

The inequality holds since f must be negative semi-definite whenever FOC identify a maximum.

Section 3

Cost Minimization

The Cost Minimization Problem

Suppose that a firm must produce y units of output.

Now, the firm chooses inputs x to minimize costs,

$$wx$$

$$\text{subject to } y = f(x).$$

The solutions are the **conditional factor or input demands**

$$x(w, y) = \begin{bmatrix} x_1(w, y) \\ x_2(w, y) \\ \vdots \\ x_n(w, y) \end{bmatrix}$$

The **cost function** is the value function of this problem

$$c(w, y) = wx(w, y) = w_1x_1(w, y) + ... + w_nx_n(w, y).$$

Cost Minimization: Solution

The Lagrangian of this program satisfies

$$L = wx + \lambda(y - f(x)).$$

If $x(w, y)$ is an interior solution and $f_i = \partial f(x)/\partial x_i$, the FOC require that

$$\frac{\partial L}{\partial \lambda} = y - f(x(w, y)) = 0,$$
$$\frac{\partial L}{\partial x_i} = w_i - \lambda f_i(x(w, y)) = 0, \quad \text{for any } i = 1, ..., n.$$

Such conditions simplify to

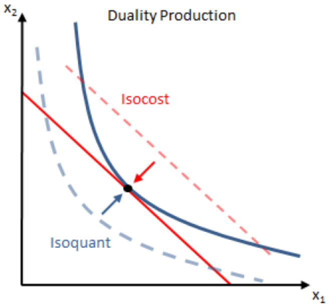
$$\frac{w_i}{w_j} = \frac{f_i}{f_j}, \text{ for any } i, j = 1, ..., n.$$

At a solution, the multiplier characterizes the marginal cost of production (i.e. marginal value of relaxing tech constraint).

Cost Minimization: Plot

For the case of two factors, $-w_1/w_2$ is the slope of the **isocost** curve

$$w_1 x_1 + w_2 x_2 = k.$$



Cost Minimization: Intuition

Why should we use cost minimization?

- it always holds locally even if the profit function is not well behaved;
- when Y is convex, there is a one-to-one mapping with profit max;
- in other words, profit max implies cost min so they do not give us more information;
- if firms are only price takers in the input market then works whereas profit max does not.

Cost Minimization: Sufficient Conditions

Consider a critical point of the Lagrangian, (x^*, λ^*),

$$f(x^*) = y \text{ and } \lambda^* f_i(x^*) = w_i \text{ for any } i = 1, ..., n.$$

If $f(x)$ is concave, x^* is a global minimum.
For local SOC, instead, study Bordered Hessian of the problem,

$$\tilde{H} = \begin{bmatrix} 0 & -f_1 & \cdots & -f_n \\ -f_1 & -\lambda f_{11} & \cdots & -\lambda f_{1n} \\ & & \vdots & \\ -f_n & -\lambda f_{n1} & \cdots & -\lambda f_{nn} \end{bmatrix}$$

If at (x^*, λ^*) the largest $n-1$ leading principal minors are negative, x^* is a strict local minimum.
[SOC for a minimization problem require the largest $n-e$ leading minors to have the sign of $(-1)^e$, where e is the number of binding constraints].

Cost Minimization: Example I

Again, consider the production function $f(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$.
FOC of the cost minimization problem require:

$$w_i - \frac{\lambda x_i^{-1/2}}{2} = 0 \text{ for } i \in \{1, 2\}.$$

Therefore, we obtain that:

$$\frac{w_1}{w_2} = \sqrt{\frac{x_2}{x_1}} \Longrightarrow \sqrt{x_2} = \sqrt{x_1}\frac{w_1}{w_2} \Longrightarrow y = \sqrt{x_1}\left(1 + \frac{w_1}{w_2}\right)$$

Consequently, conditional input demands, and costs satisfy:

$$x_i(w, y) = \left(\frac{w_j}{w_2 + w_1}\right)^2 y^2 \text{ for } j \neq i \in \{1, 2\};$$

$$c(w, y) = \frac{w_1 w_2}{w_2 + w_1} y^2.$$

Cost Minimization: Example II

Next, consider the production function $f(x_1, x_2) = \min\{ax_1, x_2\}$.

Optimality implies that $ax_1 = x_2 = y$.

Consequently, conditional input demands, and costs satisfy:

$$x_1(w, y) = \frac{y}{a};$$
$$x_2(w, y) = y;$$
$$c(w, y) = \left(\frac{w_1}{a} + w_2\right) y.$$

Cost Minimization: Example III

Finally, consider the production function $f(x_1, x_2) = ax_1 + x_2$.

By substituting the constraint with equality, we may choose x_1 to minimize

$$w_1 x_1 + w_2(y - ax_1) = (w_1 - w_2 a)x_1 + w_2 y.$$

Consequently, conditional input demands satisfy:

$$\begin{aligned}
&\text{if } w_1 < aw_2 \quad x_1(w, y) = y/a \quad \& \quad x_2(w, y) = 0 \\
&\text{if } w_1 > aw_2 \quad x_1(w, y) = 0 \quad \& \quad x_2(w, y) = y \\
&\text{if } w_1 = aw_2 \quad ax_1(w, y) + x_2(w, y) = y
\end{aligned}$$

while costs of production satisfy:

$$c(w, y) = \min\{\frac{w_1}{a}, w_2\}y.$$

These examples illustrate that **FOC approach is not ideal**. Although a solution exists by continuity, sometimes we cannot use FOC to find it.

Properties of the Cost Function

Lemma

The cost function $c(y, w)$ satisfies the following properties:

1. $c(w, y)$ is non-decreasing in w_i , for $i = 1, ..., n$;
2. $c(w, y)$ is homogeneous of degree 1 in w;
3. $c(w, y)$ is concave in w;
4. $c(w, y)$ is continuous in w.

Proof: Exercise.

The Envelope Theorem then implies the following result.

Lemma

Shephard's Lemma states that

$$\frac{\partial c(w, y)}{\partial w_i} = x_i(w, y) \text{ for any } i = 1, ..., n.$$

Properties of the Cost Function

The concavity of the cost function again implies that

$$\frac{\partial x_i(w, y)}{\partial w_i} \leq 0 \text{ for any } i = 1, ..., n.$$

By revisiting again the properties of the implied demands we can find usual conclusions about homogeneity and price effects:

- **own-price effects are negative for inputs**;
- **cross-price effects are symmetric**.

To conclude: Sometimes cost functions characterize completely technology (when technology is convex and monotone). Cost functions are an important tool, and are generally observable. The properties derived have testable implications and were derived from very few assumptions.

Section 4

Short vs Long Run Production

Short Run vs Long Run Production

Given any cost function by $c(y)$, it is possible to restate the profit maximization problem as the firm choosing y to maximize

$$py - c(y)$$

FOC for this problem require

$$p - \frac{dc(y)}{dy} = 0 \Longrightarrow \text{Price} = \text{Marginal Cost.}$$

So far we have only focused on **long run** (as all inputs were variable).

Now decompose total cost into **variable and fixed costs**:

$$c(y) = \underbrace{v(y)}_{\text{dependent on y}} + \underbrace{F}_{\text{independent of y}}$$

Fixed Costs and the Short Run

Some costs are fixed:

- because of the physical nature of inputs;
- because some inputs cannot be adjusted in the short run.

Fixed costs:

- are **sunk** and incurred even when output is zero;
- are **ignored in the short run** maximization problem.

Average Cost is defined as

$$AC(y) \equiv AVC(y) + AFC(y) \equiv \frac{v(y)}{y} + \frac{F}{y}.$$

If $AC(y)$ is increasing in y we have **diseconomies of scale**.
If $AC(y)$ is decreasing in y we have **economies of scale**.

Minimum Average Cost

First consider minimizing the average cost of production

$$\min_y AC(y) = \min_y \{ \frac{v(y)}{y} + \frac{F}{y} \}.$$

FOC for AC minimization simply require

$$\frac{dAC(y)}{dy} = \frac{1}{y} \left(\frac{dc(y)}{dy} - AC(y) \right) = 0.$$

The average cost is minimized when it coincides with the marginal cost

$$\frac{dc(y)}{dy} = AC(y).$$

The Short Run Problem

Let F be sunk in the short run.

Profits can be written as

$$py - c(y) = y(p - AVC(y)) - F$$

A firm sells only if short term profits are positive

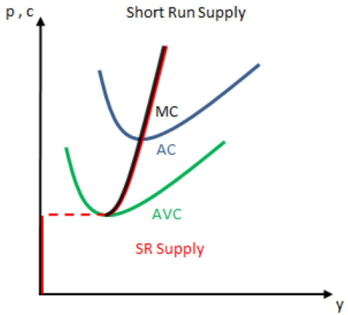
$$y(p - AVC(y)) \geq 0.$$

Profit maximization requires the firm:

- to set $y = 0$, if $p < \min_y v(y)/y$;
- to choose y so that $p = c'(y)$ otherwise.

The Short Run Supply Curve

The short run supply curve of a firm is plotted here



AC decreases before its minimum (the intersection with MC) and then increases.

AC decreases at first because both the AFC and the AVC decrease.

But AC increases once capacity is reached as AVC increases.

The Short Run Aggregate Supply

Consider a market with m firms.

Let $y_i(p)$ denote the short run supply of firm i.

The **short run industry (or aggregate) supply** is determined as follows:

$$S(p) = \sum_{i=1}^{m} y_i(p).$$

Aggregation is straightforward here.

It is, as if one big firm had made an optimal supply decision.

Production is efficiently allocated among firms as it minimizes costs!

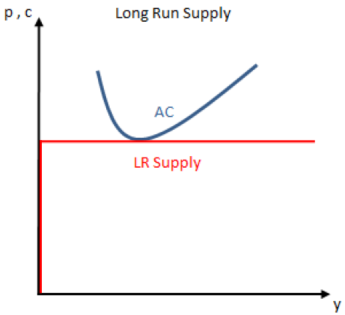
All individual properties are satisfied at the aggregate level.

These powerful results are due to price taking behavior.

Free Entry & Long Run Aggregate Supply

In the **long run**, firms can enter and exit the industry.

With no barriers to entry, the **long run industry supply is perfectly elastic** at a price equal to the **minimum average cost.**



Section 5

Markets

Producer Surplus

Consider the market for one commodity, where $y^i(p)$ is firm i's supply.

Suppose that the price of output increases from p^0 to p^1.

By Hotelling's Lemma, the change in profits for firm i is

$$\Delta^i = \int_{p^0}^{p^1} \frac{d\pi^i(p)}{dp} dp = \int_{p^0}^{p^1} y^i(p) dp.$$

We refer to Δ^i as the **producer surplus**. It identifies the area under the supply curve and measures the "willingness to pay" of a producer.

If $S(p)$ is the aggregate supply, it follows that

$$\sum_{i=1}^{m} \Delta^i = \int_{p^0}^{p^1} S(p) dp.$$

Partial Equilibrium & Elasticities

Let $D(p)$ be the **aggregate demand** for this commodity.

A **market clearing price** is a price that solves

$$S(p) = D(p).$$

A **partial equilibrium** in this market consist of a market clearing price and of the corresponding optimal demand and supply decisions.

The responsiveness of demand and supply to price changes is measured by:

- the **elasticity of demand**,

$$\epsilon^D(p) = \frac{dD(p)}{dp}\frac{p}{D(p)} = \frac{\text{\% change in demand}}{\text{\% change in price}}.$$

- the **elasticity of supply**,

$$\epsilon^S(p) = \frac{dS(p)}{dp}\frac{p}{S(p)} = \frac{\text{\% change in supply}}{\text{\% change in price}}.$$

Taxes

Consider a market in which:

- p_S denotes the price received by producers;
- p_D denotes price paid by consumers;
- t denotes the tax rate.

This setup captures both:

- **quantity taxes**, $p_D = p_S + t$;
- **value taxes** $p_D = p_S(1 + t)$.

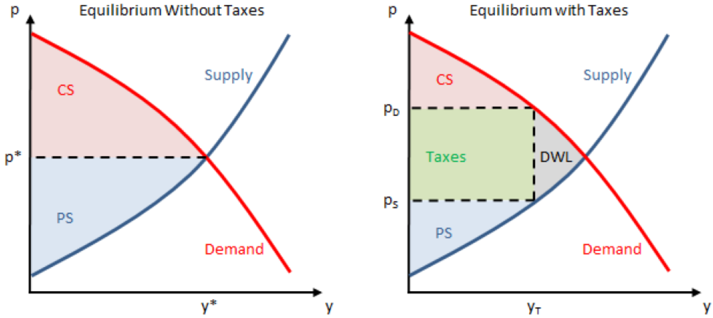
Equilibrium still requires market clearing (all units supplied are sold).

Therefore we must have that

$$D(p_D) = S(p_S).$$

Quantity Taxes

The plot depicts the equilibrium in a market with quantity taxes.



The amount that consumers and producers would want to pay for removing the tax is exceeds the revenue raised because of the **deadweight loss**.

This is the classic argument for the inefficiency of taxes.

Towards General Equilibrium

Our brief introduction to equilibrium analysis was constrained to a single market, as input prices were kept constant throughout.

However, when a price changes, prices in other markets may vary too as consumers and firms readjust consumption and production decisions.

More generally, we carried out a **partial equilibrium** analysis.

This can be misleading, as comparative statics neglect changes in other markets which may have a feedback effect on the market considered.

Next we solve for a **general equilibrium** with all the markets at once.

Section 6

Monopoly

Monoploy

If there are enough firms in a market, the influence of one single firm's decision on the market equilibrium price is very small.

If AC rises with output, a small firm is more efficient than a large one. So, no monopolization without government regulation or collusion.

If AC decreases with higher output (high entry cost: airline industry), a firm can cut prices to the point where the profit of any smaller competitor is negative.

Monopoly Problem and Solution

Monopoly problem:

$$\max \pi = \max \underbrace{p(q).q}_{R(q)} - C(q)$$

Optimal monopoly solution:

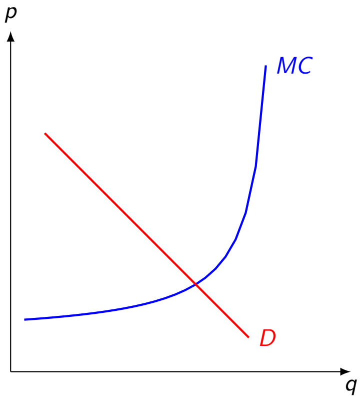
$$\frac{\partial \pi}{\partial q} = \frac{\partial R}{\partial q} - \frac{\partial C}{\partial q} = 0$$

Rewerite as:

$$MR = MC$$

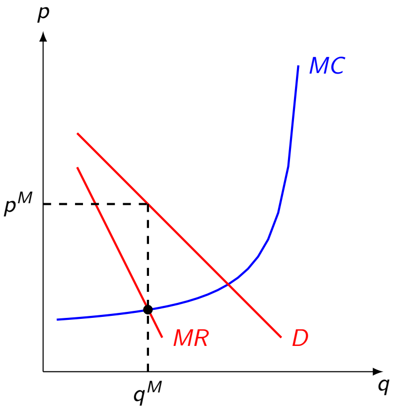
Graphical Illustration

Benefit of an additional unit = Cost of an additional unit.



Graphical Illustration

Benefit of an additional unit = Cost of an additional unit.



Markup Formula

Let ε be the elasticity of demand, then for a monpolist:

$$P\left(1+\frac{1}{\varepsilon}\right)=MC$$

A measure of degree of monopoly power is **Lerner index**:

$$\frac{P-MC}{P}=-\frac{1}{\varepsilon}$$

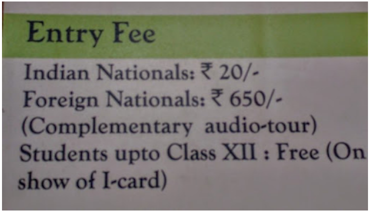
Under perfect competition: $\varepsilon=-\infty$.

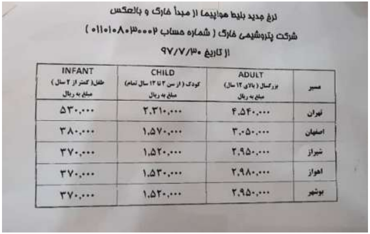
Price Discrimination

Uniform prices might be violated in practice.

If there is no arbitrage between consumers (no resale), a monopolist can potentially benefit from price discrimination:

- First Degree: (Perfect Price Discrimination)
  - The seller knows each consumer's type.
  - Prices each unit sold at the consumer's maximum willingness to pay.
- Second Degree:
  - The seller knows the distribution of consumer types only.
  - Allows the monopolist to offer consumers a quantity discount.
- Third Degree:
  - The seller knows sub-groups.
  - offers a different price for each segment of the market (or each consumer group) when membership in a segment can be observed.





نرخ جدید بلیط هواپیما از مبدأ خارک و بالعکس

شرکت پتروشیمی خارک (شماره حساب ۰۱۱۰۱۰۸۰۳۰۰۰۲)

از تاریخ ۹۷/۷/۳۰

INFANT طفل(کمتر از ۲ سال) مبلغ به ریال	CHILD کودک (از سن ۲ تا ۱۲ سال تمام) مبلغ به ریال	ADULT بزرگسال (بالای ۱۲ سال) مبلغ به ریال	مسیر
۵۳۰,۰۰۰	۲,۳۱۰,۰۰۰	۴,۵۴۰,۰۰۰	تهران
۳۸۰,۰۰۰	۱,۵۷۰,۰۰۰	۳,۰۵۰,۰۰۰	اصفهان
۳۷۰,۰۰۰	۱,۵۲۰,۰۰۰	۲,۹۵۰,۰۰۰	شیراز
۳۷۰,۰۰۰	۱,۵۳۰,۰۰۰	۲,۹۸۰,۰۰۰	اهواز
۳۷۰,۰۰۰	۱,۵۲۰,۰۰۰	۲,۹۵۰,۰۰۰	بوشهر

Third Degree PD - Basic Model

Assumptions:

- Two separate markets:

$$p_1(q_1) \quad p_2(q_2) \quad q = q_1 + q_2$$

- Cost $C(q)$

Objective function:

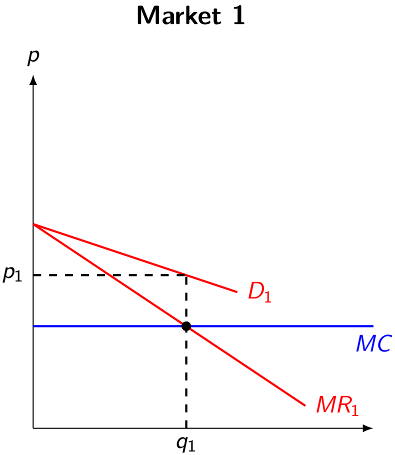
$$\max_{q_1, q_2} \pi = p_1(q_1)q_1 + p_2(q_2)q_2 - C(q)$$

FOC:

$$MR_1 = MR_2 = MC$$

Exercise: Show that the monopolist will charge a higher price for the more inelastic demand: $\varepsilon_2 > \varepsilon_1 \rightarrow p_2 > p_1$.

Graphical Illustration



First Degree Price Discrimination

It may be possible to charge each buyer the maximum price she would be willing to pay.

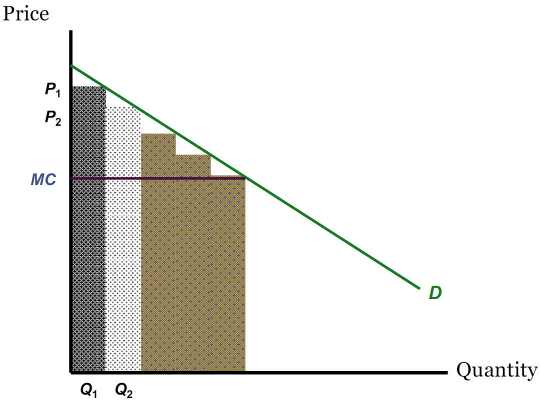
- MR curve is the same as the demand curve.
- Extracts all the consumer surplus.
- No deadweight loss.

Graphical Illustration

The first buyer pays P_1 for Q_1 units.

The second buyer pays P_2 for $Q_2 - Q_1$ units.

The monopolist will continue this way until the marginal buyer is no longer willing to pay the good's marginal cost.



Second Degree Price Discrimination

- Menu of price-quantity options that allows consumers to choose between them.
- Quantity discounts.
- Club pricing or two part tariffs.

Two-Part Tariff

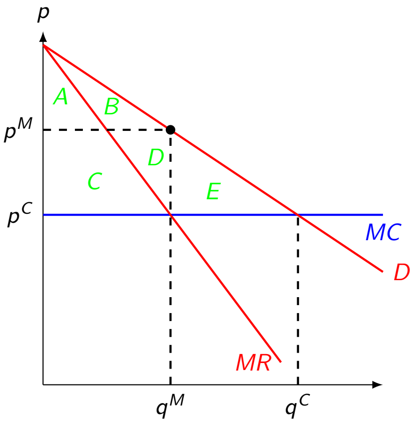
Assume that consumers are all identical in terms of the demand function.

- No entrance fee:

$$\pi = (p^M - MC)q^M \equiv \boxed{C + D}$$

- Set user fee as $P = MC$ and charge an entrance fee L equal to the total CS:

$$\pi = L + (p^C - MC)q^C$$
$$= L + 0 \equiv \boxed{A + B + C + D + E}$$



Exercise: Think of a case where consumers are non-identical. What's the best pricing strategy? Examples: Amusement parks, All you can eat restaurarnts, etc.

Additional Ways to Price Discriminate

- Bundling, or quantity discounts
  - Examples: Microsoft Office, One way vs. Round trip tickets, etc.
  - Adams and Yellen, QJE 1976.
- Intertemporal price discrimination
  - Examples: New iPhone, fashion goods, etc.
  - Lazear, AER 1986.

Section 7

New Theories of the Firm

Departures from the Classical Theory of the Firm

Several limitations of the classical theory have been addressed:

- Profit maximization may not be the true **objective of a manager**.
- **Property rights and firm ownership** may affect the equilibrium in a market with frictions. For instance, a supplier may prefer a **merger** with a client to bargaining in the market over prices and quantities, as the merger may solve incentive problems. A key prediction of Williamson's theory (validated empirically) is that firms are more likely to merge if what they produce or their assets are specific to each other and not to the rest of the market.
- **Non-profit** organizations may be successful without targeting profits. Like consumers their objectives are pinned down by preferences, and techniques extend to cover such scenarios.

Managers Objectives

The objectives of manager may differ from profits for several reasons.

Example 1: Managers may dislike effort and care only about their wage. If so, incentives (bonuses) may be required to induce the optimal effort decision. But when incentives are costly to provide, the firm might prefer induce a suboptimal effort decision.

Example 2: Managers may have career concerns. If managers are paid more in the market when they are perceived to be talented. They may care more about appearing talented and establishing a reputation, rather than about firm profits.

Reputation concerns can be beneficial and can induce managers to exert effort. However, they are occasionally harmful when managers take decisions in their best interest rather than the firm's.

Extra: Career Concerns I

Managers gain higher salaries when their reputation is high.

The state of the world w is either good or bad, $w \in \{G, B\}$.

The state is B with probability $p > 0.5$.

The correct action for the manager to take is:

- to invest in state G;
- not to invest in state B.

The manager receives a signal $s \in \{G, B\}$ about w, $Pr(s = w|w) = q$.

The signal is private information (it is why the manager is hired).

For a good manager $q = 1$, for every other manager $q = 0.5$.

The manager knows his talent, while the market believes that he is good with probability α.

Extra: Career Concerns II

Payoffs are restricted so that:

- investment is profit maximizing only if the manager is good and signal is $s = G$;
- the manager's preferences are fully determined by the updated belief α' about his talent.

If managers maximize profits the market perceives the manager:

- as good if an investment is observed, $\alpha' = 1$;
- as less likely to be good if no investment is observed, $\alpha' < \alpha$.

If manager maximize reputations:

- bad managers know that those who invest are believed to be good;
- it is not an equilibrium for them to maximize firm profits;
- bad managers to invest more than they should to show off.