School Choice

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- Heavily based on *Abdulkadiroglu, Anderson (2022)* NBER paper.
- School districts in the US and around the world are increasingly moving away from traditional neighborhood school assignment, in which pupils attend closest schools to their homes.
- Instead, they allow families to choose from schools within district boundaries. This creates a market with parental demand over publicly-supplied school seats.
- More frequently than ever, this market for school seats is cleared via market design solutions.

Matching Theory

- The question of how to design admissions in school choice programs was introduced by *Abdulkadiroglu and Sonmez (2003)*.
- As an application of matching theory *Gale and Shapley (1962)*.
- Since the introduction of the problem, economists have been deeply involved in the study and design of student assignment systems, starting with the redesign of student assignment systems in Boston and New York City.
- a two-sided many-to-one matching model
  - A student can be matched with at most one school,
  - A school can be matched with as many students as the number of available seats at the school.

Some Notions

- **Feasiblity**: A matching of students and colleges is is said to be feasible if each student is matched with at most one college and the number of students matched with a college does not exceed the capacity of the college.
- **Stability**: If a student and a college are matched, the student prefers the college to remaining unassigned and the college prefers filling one of its seats with the student to leaving the seat empty.
- **Justified Envy**: A matching of students and schools is said to be free of justified envy if whenever a student prefers a school to her match, either she is not eligible, or the school is already fully matched with students that the school ranks higher.

School Choice Problem

- A set of students: $N = \{1, ..., |N|\}$.
- A set of schools: $S = \{s_1, ..., s_{|S|}\}$.
- The number of available seats at school s is denoted by q_s.
- Each student $i \in N$ has strict preferences P_i over schools and being unmatched, denoted $S \cup \{i\}$, where $\{i\}$ represents being unmatched for student i.
- Each school $s \in S$ has a weak relation $\succeq_s$ over $N \cup \{s\}$, where $\{s\}$ represents keeping a seat empty.

Some Considerations

- It is assumed that each student ranks schools without any regard to enrolled students at schools.
- Schools may have more complicated preferences over sets of students that may not be captured by a simple ranking of individual students.
- When school rankings are strict and reflects school preferences, the model reduces to that of *Gale and Shapley (1962)*.

Matching

- If μ denotes a matching of students and schools, $\mu(a)$ is the match of $a \in N \cup S$,
  - Each student $i \in N$ is assigned a school or remains unmatched, i.e. $\mu(i) \in S \cup \{i\}$,
  - Each school $s \in S$ is matched with a set of students up to its capacity, i.e. $\mu(s) \subset N$
  - And $|\mu(s)| \leq q_s$,
  - And $\mu(i) = s \in S$ if and only if $i \in \mu(s)$.
- A **mechanism** φ determines a matching for any given problem $(N, S, q, P, \succeq)$.

Policy Objectives

- Three major policy objectives that have been critical in the design of real-life school admissions procedures, namely - Efficiency,
  - Stability,
  - Strategy-proofness.

Pareto Efficiency

- A matching μ Pareto dominates another matching μ' if every student weakly prefers μ to μ', i.e.

$$\mu(i) R_i \mu'(i) \ \text{ for all } \ i \in N;$$

and at least one student strictly prefers μ to μ',

$$\mu(i) P_i \mu'(i) \ \text{ for some } \ i \in N;$$

Stability

- The stability notion *Gale and Shapley (1962)*
  - Matching μ is individually rational if every student is matched with an acceptable school at which she eligible or remains unmatched.
  - Cannot be blocked by any student-school pair. A student-school pair blocks μ if they mutually prefer to be matched to each other.
- *Roth (2002)* shows that stable matching is key for long term survival of centralized markets in the entry level labor markets.
- There is a fundamental trade-off between stability and Pareto efficiency.

Example 1 - Roth (1982)

- There are three students 1, 2, and 3, and three schools, s_1, s_2, and s_3. Each school has one available seat.
- Student preferences and school rankings are given by:

$$
\begin{array}{ll}
1 : s_2 P_1 s_1 & s_1 : 1 \succ_{s_1} 3 \succ_{s_1} 2 \\
2 : s_1 P_2 s_2 P_2 s_3 & s_2 : 2 \succ_{s_2} 1 \succ_{s_2} 3 \\
3 : s_1 P_3 s_2 P_3 s_3 & s_3 : 2 \succ_{s_3} 3
\end{array}
$$

Example 1 Cont.

- In this problem there is a unique stable matching:

$$\mu = ((1, s_1), (2, s_2), (3, s_3)).$$

- The stable matching μ is Pareto dominated by the following Pareto efficient matching:

$$\mu' = ((1, s_2), (2, s_1), (3, s_3)).$$

- Students 1 and 2 are matched to their first choices under μ'.
- However, $(3, s_1)$ forms a blocking pair because 3 prefers s_1 to s_3, and s_1 ranks 3 higher than 2.

Strategy-Proofness

- A matching mechanism is strategy-proof if it is *dominant strategy incentive compatible* for all strategic participants, which includes all students in N and all strategic schools in S.
- Dominant strategy incentive compatibility ensures that each strategic agent finds reporting true preferences to the mechanism as best strategy regardless of what the agent knows about the game and regardless of how other agents act in the game.

Matching Algorithms

1. The Deferred Acceptance Algorithm
2. The Immediate Acceptance Algorithm
3. The Top Trading Cycles Algorithm
4. The Serial Dictatorship Mechanism

Deferred Acceptance

- *Step 1.* Every student i applies to her most preferred school according to her preferences $\succ_i$. Every school s considers the students applying to it, and rejects ineligible students and **provisionally assigns** its seats to the remaining applicants in the order of its ranking $\succ_s$. When all seats at s are provisionally assigned, the school rejects all the remaining students.
- *Step k.* Every student i that is rejected in the previous step applies to her next preferred school in $\succ_i$. Every school s considers students that it has provisionally assigned a seat in the previous step and students that apply in this step. From this set, school s rejects ineligible students and provisionally assigns its seats to the remaining students in the order of its ranking $\succ_s$. When all seats at s are provisionally assigned, the school rejects the remaining students.

Deferred Acceptance

- The algorithm has a long history in entry level labor markets in medicine and law (Roth, 2008).
- **Theorem**: *(Gale and Shapley, 1962)* The deferred acceptance algorithm converges to a stable matching in a finite number of steps.

Example 2 - DA

$$1 : s_2 P_1 s_1 P_1 s_3 \qquad s_1 : 1 \succ_{s_1} 3 \succ_{s_1} 2$$

$$2 : s_1 P_2 s_2 P_2 s_3 \qquad s_2 : 3 \succ_{s_2} 1 \succ_{s_2} 2$$

$$3 : s_1 P_3 s_2 P_3 s_3 \qquad s_3 : 1 \succ_{s_3} 2 \succ_{s_3} 3$$

- In step 1, 1 is *provisionally* assigned the single seat at s_2. Since $3 \succ_{s_1} 2$, s_2 provisionally assigns its single seat to 3 and rejects 2.
- In the second step, 2, the only student that was rejected in Step 1, applies to her next most preferred school, s_2. Since the school has a single seat and $1 \succ_{s_2} 2$, s_2 provisionally assigns student 1 and rejects student 2.
- In third step, 2 applies to her next most preferred school, s_3 and the provisional assignments are finalized.

Example 2 Cont.

- The matching is given by

$$\mu = ((1, s_2), (2, s_3), (3, s_1))$$

- The roles of students and schools may be swapped. In that version of the algorithm, schools make offers to students, each student keeps the best among all offers she receives and rejects the remaining offers. That yields

$$\mu = ((1, s_1), (2, s_3), (3, s_2))$$

- The former version of the deferred acceptance algorithm is called *student-optimal* and the latter is called *school-optimal*.

Immidiate Acceptance

- *Step 1.* Every student i applies to her most preferred acceptable school in $\succ_i$. Every school s **permanently assigns** its seats to its eligible applicants in the order of its ranking $\succ_s$. It rejects ineligible applicants. When all seats are permanently assigned, the school rejects the remaining applicants.
- *Step k.* Every student i who was rejected in the previous step applies to her k^{th} most preferred acceptable school. Every school s with available seats permanently assigns its remaining seats to its new eligible applicants in the order of $\succ_s$. It rejects ineligible applicants. When all seats are permanently assigned, the school rejects all the remaining applicants.

Immidiate Acceptance

- It places more students to their most preferred schools than other mechanisms.
- **Theorem**: *(Abdulkadiroglu and Sonmez, 2003)* The immediate acceptance algorithm converges in a finite number of steps to a matching that is Pareto efficient with respect to P_N.
- Pareto efficiency of the immediate acceptance algorithm is with respect to submitted preferences, as is the stability of the deferred acceptance algorithm.
- algorithm may punish families that submit their preferences truthfully.

Example 3 - IA

$$1 : s_2 P_1 s_1 P_1 s_3 \qquad s_1 : 1 \succ_{s_1} 3 \succ_{s_1} 2$$
$$2 : s_1 P_2 s_2 P_2 s_3 \qquad s_2 : 2 \succ_{s_2} 1 \succ_{s_2} 3$$
$$3 : s_1 P_3 s_2 P_3 s_3 \qquad s_3 : 2 \succ_{s_3} 1 \succ_{s_3} 3$$

- In step 1, 1 is the only applicant at s_2, 1 is permanently assigned the single seat at s_2. 2 and 3 apply to s_1 in this step. Since $3 \succ_{s_1} 2$, student 3 is assigned the single seat at s_1 and 2 is rejected by the school.
- In the second step, 2 applies to her second most preferred school s_2. There are no remaining seats at s_2, so 2 is rejected by s_2.
- Then 2 applies to her third most preferred choice s_3 in Step 3. She is permanently assigned the single seat at s_3.

Example 3 Cont.

- The outcome of the immediate acceptance algorithm is given by

$$\mu^{IA} = ((1, s_2), (2, s_3), (3, s_1))$$

- The unique stable matching, and therefore the outcome of both versions of the deferred acceptance algorithm, is

$$\mu^{DA} = ((1, s_1), (2, s_2), (3, s_3)).$$

- The deferred acceptance and the immediate acceptance algorithm need not recommend the same matching!

Top Trading Cycles

- First investigated by *Shapley and Scarf (1974)*.
    - House allocation problem in which each economic agent owns a house and would like to swap it for a more preferred option.
- **Theorem**: *(Abdulkadiroglu and Sonmez, 2003)* The top trading cycles algorithm converges in a finite number of steps to a Pareto efficient matching with respect to P_N.

Top Trading Cycles Algorithm

- *Step 0.* Every student and every school are initially available.
- *Step k.* An available student becomes unavailable when she is assigned, or if none of the available schools at which she is eligible are acceptable for her. In the latter case, she remains unassigned. An available school becomes unavailable when all of its seats are assigned, or if none of the available students that find the school acceptable are eligible at the school. In the latter case, the remaining seats at the school remain unfilled.

Top Trading Cycles Algorithm Cont.

- Every available student i points to her most preferred acceptable school among all available ones. Every available school s points to the eligible student that is highest ranked in $\succ_s$ among all available students.
- A cycle is an ordered list of students and schools $(i_1, s_1, i_2, s_2, ..., i_n, s_n)$ such that, for each $k = 1, ..., n$, student i_k points to school s_k and school s_k points to student i_{k+1}, where $n + 1$ is replaced by 1.
- For each cycle, assign each student in the cycle to a seat at the school that she points to.
- The algorithm terminates if there are no available students or no available schools.

Example 4 - TTC

$$1 : s_2 P_1 s_1 P_1 s_3 \qquad s_1 : 1 \succ_{s_1} 3 \succ_{s_1} 2$$

$$2 : s_1 P_2 s_2 P_2 s_3 \qquad s_2 : 2 \succ_{s_2} 1 \succ_{s_2} 3$$

$$3 : s_1 P_3 s_2 P_3 s_3 \qquad s_3 : 2 \succ_{s_3} 1 \succ_{s_3} 3$$

- In the beginning of Step 1, all students and school are available.
- Student 1 points to s_2, 2 and 3 point to s_1. School s_1 points to 1, s_2 and s_3 point to 2.
- There is a cycle in which 1 points to s_2, s_2 points to 2, 2 points to s_1 and s_1 points to 1. Student 1 is assigned the single seat at her first choice s_2 and 2 is assigned the single seat at his first choice s_1. Students 1 and 2, as well as schools s_1 and s_2, become unavailable.
- In Step 2, the only available student is 3 and the only available school is s_3. They point to each other, forming a cycle, and student 3 is assigned her last choice s_3.

Example 4 Cont.

- The outcome of the top trading cycles algorithm is:

$$\mu^{TTC} = ((1, s_2), (2, s_1), (3, s_3))$$

- Note that for the preferences given in Example 3, this solution is different from both the outcome of the deferred acceptance algorithm and the immediate acceptance algorithm.

Serial Dictatorship

- First analyzed for queuing problems, such as assigning individuals to offices or public housing depending on their positions in a queue or waiting list.
- It only needs student preferences, and a queue or ordering of students.
- The serial dictatorship algorithm can also be implemented via the deferred acceptance and top trading cycles algorithms.

The Algorithm

- Given a strict ordering of students,

  Step k. The k-th student in the ordering is assigned a seat at her most preferred acceptable school among all schools with available seats, at which she is also eligible. If no such school exists, she remains unassigned.

  The algorithm terminates when all students in the ordering are processed.

Efficiency

- **Theorem**: If all school rankings are set to the ordering used in the serial dictatorship, then the outcomes of deferred acceptance, top trading cycles and serial dictatorship algorithms are the same for every P_N.
- **Corollary:** The serial dictatorship mechanism is Pareto efficient with respect to every P_N.
- Notice that the deferred acceptance algorithm also becomes Pareto efficient when schools share the same ranking.